Understanding HANK: Insights from a PRANK*

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Abstract

We show analytically that whether incomplete markets resolve New Keynesian 'paradoxes' depends primarily on the cyclicality of income risk, rather than marginal propensity to consume (MPC) heterogeneity. Incomplete markets reduce the effectiveness of forward guidance and multipliers in a liquidity trap only with procyclical risk. Countercyclical risk amplifies these 'puzzles'. Procyclical risk permits determinacy under a peg; countercyclical risk generates indeterminacy even under the Taylor principle. MPC heterogeneity leaves determinacy and paradoxes qualitatively unaffected, but can change the sensitivity of GDP to interest rates. By affecting the cyclicality of risk, even 'passive' fiscal policy influences the effects of monetary policy.

Keywords: New Keynesian, incomplete markets, monetary and fiscal policy, determinacy, forward guidance, fiscal multipliers

JEL codes: E21, E30, E52, E62, E63

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1 Introduction

The last few years have seen a surge of interest in heterogeneous agent New Keynesian (HANK) models. Heterogeneity and market incompleteness have been proposed as a means to understand the monetary transmission mechanism (Kaplan et al., 2016), the forward guidance puzzle (McKay et al., 2015), the distributional effects of monetary policy (Gornemann et al., 2016), the efficacy of targeted transfers (Oh and Reis, 2012), automatic stabilizers (McKay and Reis, 2016b) and fiscal stimulus (Hagedorn et al., 2017), among many other topics. These explorations have revealed that the introduction of market incompleteness into the NK model can affect not just the model's substantive predictions, but also the determinacy properties of equilibrium (Ravn and Sterk, 2017b; Auclert et al., 2017), which are central to fundamental questions in monetary economics - how is the price level determined, and what kind of policy regime ensures price stability? But the source of the differences between HANK and representative agent New Keynesian (RANK) economies, and the extent to which these differences are a general result rather than a consequence of particular modeling assumptions, remain obscure. This is because incomplete market models are generally analytically intractable since the distribution of wealth is an infinite dimensional state variable; thus, most of these studies make use of computational methods. While these papers have highlighted striking differences in the behavior of HANK and RANK economies, the lack of analytical tractability makes it hard to to drill down and uncover exactly which features are responsible for these differences.

In particular, while the HANK literature has emphasized the importance of marginal propensity to consume (MPC) heterogeneity and precautionary savings in understanding how HANK economies differ from RANK, the precise role played by each remains unclear. While Werning (2015) argues that the cyclicality of risk, rather than the precautionary savings motive per se, affects whether and how HANK differs from RANK, Bilbiie (2017a) has argued it is the cyclicality of the income of high MPC individuals which is key to understanding these differences. In this paper we aim to distinguish the *distinct* effects of precautionary savings and the cyclicality of risk, on the one hand, and heterogeneity in households' MPCs and the cyclical sensitivities of their incomes, on the other hand, in HANK economies. In other words, which matters more in explaining how HANK economies differ from RANK - the change in behavior of driven by the variable risk they face over the business cycle or who gains and who loses in a recession? We present an analytically tractable HANK model to answer this question.

To isolate the distinct effects of precautionary savings and MPC heterogeneity, we begin with an economy in which the latter is absent. This benchmark is a standard NK economy, with the exception that individuals face idiosyncratic, uninsurable shocks to their endowment of labor. Importantly, idiosyncratic income risk varies endogenously with aggregate economic activity, and may be either procyclical or countercyclical. The economy permits closed-form solutions because household utility has constant absolute risk aversion (CARA), rather than constant relative risk aversion (CRRA). This permits linear aggregation and gives us an exact closed form aggregate Euler equation, without having to carry around an infinite-dimensional state variable or impose a degenerate wealth distribution. Thus, one can think of it as a Pseudo-Representative Agent New-Keynesian model - in short, a PRANK. Again, our goal is to understand the qualitative differences between HANK and RANK economies and is *<u>not</u>* quantitative in nature.

Even absent MPC heterogeneity, uninsurable income risk can dramatically alter the properties of a NK economy, in a way which critically depends on the cyclicality of income risk. Firstly, market incompleteness can alter the determinacy properties of equilibrium. RANK models feature indeterminacy under an interest rate peg, or more generally, under interest rate rules which fail to satisfy the Taylor principle. HANK models can feature determinacy under a peg - but only when income risk is procyclical. In this case, even under a peg, higher future output and inflation cannot be self-fulfilling, because it would also imply higher income risk, reducing demand via the precautionary savings channel. Whereas procyclical income risk makes indeterminacy less likely, countercyclical risk makes it more likely - if risk is countercyclical, the standard Taylor principle may not even be sufficient to ensure determinacy. In this case, fear of lower output in the future implies higher risk, depressing demand via the precautionary savings channel and generating a self-fulfilling recession. This clarifies that Ravn and Sterk (2017b)'s finding that incomplete markets make determinacy less likely is driven by the fact that risk is countercyclical in their model. We derive a general, income-risk augmented Taylor principle which depends explicitly on the cyclicality of income risk.

Importantly, the cyclicality of income risk is endogenous. In particular, it depends on the cyclicality of fiscal policy, and on whether redistribution increases or decreases when output is low. This highlights a new and important dimension of monetary-fiscal interaction, distinct from (but related to) the traditional question concerning whether the fiscal authority adjusts surpluses in order to repay government debt along any hypothetical price path (Leeper, 1991). In HANK economies, what matters is not just the expected path of surpluses, but whether those surpluses are raised in ways that increase or decrease the variance of households' after-tax income, and whether this depends on the overall level of economic activity.

A recent literature has argued that incomplete markets can solve "puzzles" arising in RANK models. We show next that whether incomplete markets can or cannot solve these puzzles again depends crucially on the cyclicality of income risk. In RANK models, announcements of future interest rate cuts are equally, or more, effective than current policy changes in stimulating output and inflation. Market incompleteness can reverse this prediction, but only if income risk is strongly procyclical, so the expansionary effect of a promised future boom is offset by an increase in desired precautionary savings in response to the increased risk generated by the boom. If risk is countercyclical, this prediction is reversed, and incomplete markets *worsen* the 'forward guidance puzzle' (FGP). Interestingly, HANK models may feature a stronger forward guidance puzzle than RANK even if income risk is acyclical or weakly procyclical. Looser monetary policy effectively provides more consumption insurance against income shocks, reducing consumption risk (which is ultimately what matters for precautionary savings) for a given level of income risk, and boosting demand.

Next, we examine how incomplete markets alter another controversial prediction of RANK models. RANK models predict that in a liquidity trap, the government spending multiplier is greater than 1 and increasing in the duration of the trap.[1] This is due to the expected inflation channel: when nominal interest rates are constrained due to the zero bound, higher future spending increases expected inflation, lowers real interest rates, and stimulates current spending. If income risk is procyclical, the precautionary savings channel can potentially outweigh the effect of expected inflation in our HANK economy. While future spending lowers real interest rates, it also increases risk, encourages households to save, and moderates the increase in current spending. Consequently, the multiplier can be *less* than 1 and *decreasing* in the

[1] This prediction depends on the assumption that the monetary authority targets the zero-output, zero-inflation steady state (using an appropriately specified active Taylor rule) as soon as the underlying shocks abate and the ZLB is no longer binding. We maintain this assumption throughout our analysis. Cochrane (2017b) discusses how fiscal multipliers change when the active Taylor rule assumption is dropped and alternative criteria are used to select among the many bounded rational expectations equilibria consistent with a given path of nominal interest rates.

duration of the liquidity trap. In contrast, if risk is countercyclical, the precautionary savings and expected inflation channels both work in the same direction, increasing the multiplier.

The results just described reveal that, even absent MPC heterogeneity, the precautionary savings motive can dramatically change outcomes in a NK economy - albeit in a way which critically depends on the cyclicality of income risk. This might seem surprising since several authors have argued that MPC heterogeneity (Kaplan et al., 2016) and in particular the cyclicality of high MPC agents' income (Bilbiie, 2017a) are crucial in understanding how HANK economies differ from RANK. One reason for this confusion is that both the existing analytical and computational literatures have found it hard to distinguish these features from precautionary savings and the cyclicality of income risk. To study MPC heterogeneity we introduce a fraction of hand-to-mouth (HTM) individuals (Campbell and Mankiw, 1989) into our benchmark economy. HTM income may be more or less cyclically sensitive relative to the income of unconstrained households; unlike Werning (2015) and Bilbiie (2017a), our model imposes no restriction between this cyclical sensitivity and the cyclicality of income risk. This allows us to clarify that MPC heterogeneity has an important, but distinct, role in affecting aggregate outcomes relative to the cyclicality of income risk.

MPC heterogeneity does not affect the determinacy properties of equilibrium, and cannot solve the FGP on its own.[2] However, it can increase or decrease (depending on cyclical sensitivity of HTM income) the contemporaneous sensitivity to current interest rates, which the precautionary channel left untouched. Werning (2015) argues that the cyclicality of income risk affects the sensitivity of aggregate consumption to current and future interest rates, while Bilbiie (2017a) argues instead that the cyclical sensitivity of HTM income is key; but both authors work with models which impose a tight connection between these two forces. Our framework makes clear that the sensitivity of consumption to current interest rates is wholly governed by the cyclical sensitivity of HTM income, while the relative sensitivity of consumption to current and future rates is determined by the cyclicality of risk.

Related Literature Like us, some other recent papers have also made simplifying assumptions in order to solve HANK models analytically in order to better understand the operative channels. Ravn and Sterk (2017b); Bilbiie (2017a); Werning (2015) and Challe (2017) assume that agents are unable to borrow and the government issues no debt - the so called *zero liquidity limit*.[3] This assumption makes the wealth distribution degenerate, affording analytical tractability. While a useful simplifying assumption, this has the strong implication that income risk passes through one for one to consumption risk. In reality, households can partially insure consumption against income shocks through various mechanisms (Blundell et al., 2008); thus the pass-through from income to consumption risk is less than one, and importantly, may vary over time. Our approach does not impose zero liquidity, and allows for endogenous, time-varying pass-through of income to consumption risk, an important and as yet understudied component of the precautionary savings channel. To be clear, this important component of the precautionary savings channel is already implicitly present in models solved using computational methods; the advantage of our approach is that we can observe it analytically.

The zero liquidity limit papers just described also impose a tight connection between the cyclicality of

[2] While cyclicality of income risk is the most important role in determining the size of the balanced budget government spending multiplier, MPC heterogeniety also plays a subtle role as we discuss in Section 6.

[3] Werning (2015) also considers models which relax the zero liquidity limit.

income risk and the cyclical sensitivity of HTM income. Intuitively, suppose individual income can either be high or low, and high income agents are unconstrained but face a constant probability of becoming low income and constrained. Then the risk faced by unconstrained agents will be higher in recessions only if the gap between high and low income is higher in recessions, i.e. the income of constrained agents is more cyclically sensitive. As described previously, this tight connection leads Werning (2015) to argue that the cyclicality of risk governs the sensitivity of aggregate consumption to current and future interest rates, while Bilbiie (2017a) makes the same claim about the cyclicality of HTM income: we delineate the distinct roles of each.

Another difference, relative to Werning (2015), is that we discuss how the cyclicality of income risk affects determinacy in HANK models, not just the equilibrium response of consumption to interest rates. In this regard our results are related to Auclert et al. (2017), who also analyze how incomplete markets affects determinacy and the economy's response to increases in government spending, monetary policy shocks and forward guidance.[4] Their analytical results are framed in terms of an infinite dimensional $\mathbf{M}$ matrix which describes the response of consumption at any date to aggregate output at any other date; for example, they show that determinacy depends on the asymptotic properties of the far-out columns of this matrix. They also present numerical results which generally confirm the results in our closed-form solutions (procyclical risk permits determinacy under a peg, countercyclical risk makes determinacy less likely, and so forth). Our simplifying assumption of CARA utility allows us to analyze determinacy and the economy's response to shocks in a transparent model permitting closed form solutions.[5]

Recent work by Bilbiie (2017b); Debortoli and Galí (2017) presents a TANK (*two agent New Keynesian*) model to shed light on how the responses of HANK models differs from RANK models in response to aggregate shocks. Similarly, Bilbiie (2008) discusses determinacy in a TANK model, while Mehrotra (2017) compares the effects of transfers and government purchases. As Debortoli and Galí (2017) emphasize, this TANK literature abstracts from precautionary savings (more generally, heterogeneity *within* unconstrained households) in order to study MPC heterogeneity (heterogeneity *between* constrained and unconstrained households). Our approach permits a discussion of both the precautionary savings channel and MPC heterogeneity in detail.[6]

McKay et al. (2015) argued that incomplete markets solve the "forward guidance puzzle" (Del Negro et al., 2015), i.e. the fact that in NK models, announcements of interest rate cuts far in the future are more effective at stimulating output and inflation than contemporaneous interest rate cuts . (McKay et al., 2017) present a stylized incomplete markets model, again with zero liquidity, in which household consumption is described by a 'discounted Euler equation'. We also derive a modified Euler equation in our CARA-HANK framework (which does not rely on zero liquidity) and describe the conditions under which forward guidance is less effective than in a RANK model. Importantly though, we find that the model only generates a *discounted* Euler equation and weakens the power of forward guidance if income risk is sufficiently procyclical (as in McKay et al. (2017)).[7] If instead income risk is countercyclical, the

[4]Indeed, this paper began life as a discussion of Auclert et al. (2017).

[5]Bilbiie (2017a) also discusses determinacy properties but as mentioned above, his framework makes it hard to distinguish the roles played by the cyclicality of income risk and the cyclical sensitivity of HTM income.

[6]In older work, Challe and Ragot (2011); Challe et al. (2017) and Challe et al. (2017) make assumptions on preferences, technology and market structure in order to construct analytically tractable *limited heterogeneity equilibria* in which the wealth distribution has finite support. These papers primarily study how the precautionary savings channel can amplify aggregate shocks, which is related to, but distinct from, the themes we address in this paper.

[7]However, it is not necessary that agents expect to be off their Euler equation in the future.

model generates an *explosive* Euler equation and strengthens the power of forward guidance.

The rest of the paper is structured as follows. Section 2 presents the model economy. Section 3 solves the model and discusses the factors affecting the cyclicality of income risk. Section 4 shows how the cyclicality of risk affects determinacy of equilibrium in our HANK economy and derives an income risk-adjusted Taylor principle. Section 5 discusses conditions under which the introduction of incomplete markets solves, or amplifies, two perceived 'puzzles' present in the RANK model: the power of forward guidance, and explosive government spending multipliers in a liquidity trap. Section 6 discusses the relative importance of hand-to-mouth agents and the precautionary saving motive in HANK economies. 7 discusses the pervasive importance of fiscal policy. Section 8 concludes.

2 Model

We introduce uninsurable income risk into an otherwise standard New Keynesian model. Households face idiosyncratic income risk and can only save in a nominally riskless bond. The supply side is deliberately kept relatively standard: monopolistically competitive firms combine labor and intermediate inputs to produce differentiated varieties of the output good, and set prices subject to nominal rigidities. For simplicity, we consider an economy with idiosyncratic risk but no aggregate uncertainty.

2.1 Households

There is a continuum of households in the economy indexed by $i \in [0,1]$ who solve:

$$\begin{aligned} \max \qquad & -\frac{1}{\gamma}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{-\gamma c_t^i} \\ \text{subject to} \qquad & P_t c_t^i + \frac{1}{1+i_t} A_{t+1}^i = A_t^i + P_t y_t^i \end{aligned}$$

Each household can save only in a risk free nominal bond A_{t+1}^i which has a price of $\frac{1}{1+i_t}$ at date t and pays off 1 in nominal terms at $t+1$. c_t^i is itself an aggregate consumption index defined by $c_t^i = \left[\int_0^1 c_t^i(k)^{\frac{\theta-1}{\theta}} dk\right]^{\frac{\theta}{\theta-1}}$ As is standard, the demand for variety k by household i can be written as $c_t^i(k) = \left(\frac{P_t(k)}{P_t}\right)^{-\theta} c_t^i$. Thus, total consumption demand for variety k can be written as $c_t(k) = \int_0^1 c_t^i(k)dk = \left(\frac{P_t(k)}{P_t}\right)^{-\theta} c_t$ where $c_t = \int_0^1 c_t^i di$.

Income of Household i $\quad y_t^i$ denotes the income of household i in period t and can be written as:

$$y_t^i = (1-\tau_t)\,\omega_t \ell_t^i + D_t^i + T_t \tag{1}$$

The income of each household is made up of three components: (i) real labor income net of taxes $(1-\tau_t)\,\omega_t \ell_t^i$, (ii) real dividends from the production sector, D_t^i and (iii) real transfers from the government, T_t. We discuss each of these subcomponents next.

Labor Income Following Aiyagari (1994), we assume that households have a stochastic endowment of labor ℓ_t^i each period which they supply inelastically at the prevailing real wage ω_t. In particular, we assume that each period, household i's endowment of labor is given by $\ell_t^i \sim N\left(\ell, \sigma_{\ell,t}^2\right)$ where ℓ is the aggregate endowment of labor in this economy. Without loss of generality, we normalize $\ell = 1$. τ_t denotes the linear tax on labor income. In particular we assume that $\sigma_{\ell,t}^2$ is given by:

$$\sigma_{\ell,t}^2 = \boldsymbol{\sigma}_{\boldsymbol{\ell}}^2(Y_t) \tag{2}$$

where Y_t denotes aggregate output. As in McKay and Reis (2016a), this specification allows for cyclical changes in the distributions of earnings risks in line with the empirical evidence documented by Storesletten et al. (2004) and Guvenen et al. (2014). To be clear, none of our results depend on the assumption that the variance of labor endowments depends exogenously on economic activity. Even if the variance of endowments $\boldsymbol{\sigma}_{\boldsymbol{\ell}}^2$ does not vary with economic activity and is constant, the variance of household income will generally still vary with economic activity, as we show in Section 3.4.

Capital Income In addition to labor income, each household also receives dividends from the productive sector. Notice that the dividends D_t^i have an i superscript, implying that dividends may vary across households. The dividends received by household i are:

$$D_t^i = \overline{D}_t + \delta_t\left(\ell_{i,t} - 1\right) \tag{3}$$

As has been highlighted by Broer et al. (2016) and Werning (2015), the distribution of dividends is an important determinant of how an incomplete markets economy responds to various shocks. This convenient specification is fairly general and nests many commonly used cases. For example, $\delta = 0$ implies that dividends are distributed equally across all households; $\delta > 0$ implies that households with larger labor income are the recipient of a larger share of dividends. We also allow for the possibility that δ_t varies with economic activity: $\delta_t = \boldsymbol{\delta}(Y_t)$.

Net Transfers from the Government The last source of income is lump sum transfers net of taxes. We assume the government makes a lump sum transfer T_t which is the same across all households in each period, and taxes labor income at the rate τ_t.

2.2 Firms

There is a continuum of monopolistically competitive firms indexed by $j \in [0, 1]$. Following Basu (1995); Woodford (2003); Nakamura and Steinsson (2010), we assume that each firm combines labor and intermediate inputs to produce a differentiated good $x(j)$ using a constant returns to scale technology:[8]

$$x_t(j) = M_t(j)^{\alpha} L_t(j)^{1-\alpha} \tag{4}$$

[8] In our setup, since households supply their stochastic endowment of labor inelastically, intermediate inputs are the factor which adjusts in response to demand. This specification of the supply side of the economy is just one possibility among many. Our results are general enough to apply to any specification of the supply side which yields a New Keynesian Phillips curve relationship - with the understanding that different specifications of the supply side may affect whether income risk is procyclical or countercyclical.

where $M_t(j)$ is the level of intermediate inputs utilized by the firm producing variety j. $M_t(j)$ is itself an aggregate of intermediate inputs defined by $M_t(j) = \left[\int_0^1 m_t(j,k)^{\frac{\theta-1}{\theta}} dk\right]^{\frac{\theta}{\theta-1}}$. As is standard, the demand for intermediate input k by firm j can be written as $m_t(j,k) = \left(\frac{P_t(k)}{P_t}\right)^{-\theta} M_t(j)$. Thus, total demand for intermediate input k can be written as $m_t(k) = \int_0^1 m_t(j,k)dj = \left(\frac{P_t(k)}{P_t}\right)^{-\theta} M_t$. Firms solve the cost minimization problem

$$\min P_t M_t(j) + W_t L_t(j) \qquad \text{subject to} \qquad M_t(j)^{\alpha} L_t(j)^{1-\alpha} \geq x_t(j)$$

yielding $\frac{W_t}{P_t} = \frac{1-\alpha}{\alpha} M_t(j)$.

In symmetric equilibrium, $x_t(j) = x_t$. Normalizing the aggregate endowment of labor to 1, we have $L_t(j) = 1$ and so real wages are given by

$$\omega_t = \frac{W_t}{P_t} = \frac{1-\alpha}{\alpha} x_t^{\frac{1}{\alpha}} \tag{5}$$

Finally, net output is given by

$$Y_t = x_t - M_t = x_t - x_t^{\frac{1}{\alpha}} \tag{6}$$

2.3 Nominal Rigidities

Each firm faces a quadratic cost of changing prices following Rotemberg (1982). The pricing decisions of each firm can then be written as:

$$\max_{P_t(k)} \sum_{t=0}^{\infty} \frac{1}{\prod_{s=0}^{t-1}(1+r_s)} \left\{ \left(\frac{P_t(k)}{P_t} - \lambda_t\right) \left(\frac{P_t(k)}{P_t}\right)^{-\theta} - \frac{\Psi}{2} \left(\frac{P_t(k)}{P_{t-1}(k)} - 1\right)^2 \right\} x_t$$

where $\lambda_t = \frac{\omega_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$ is the real marginal cost faced by firm k, $1 + r_t = \frac{1+i_t}{1+\Pi_{t+1}}$ denotes the real interest rate. $\frac{\Psi}{2}\left(\frac{P_t(k)}{P_{t-1}(k)} - 1\right)^2 x_t$ denotes the quadratic cost a firm faces if it wants to change its price from last period's level. Following Ascari and Rossi (2012) and Bhandari et al. (2017) we assume that this cost is rebated lump sum to households along with dividends.[9] $\Psi \geq 0$ is a constant which scales the cost.

2.4 Policy

Monetary Policy We assume that the monetary authority sets nominal rates according to some rule:

$$i_t = (1+r)\Pi_t^{\Phi_\pi} \geq 0 \tag{7}$$

where $(1+r)$ denotes the steady state real interest rate.

[9] This assumption is made to simplify exposition. Even if we did not rebate this cost to households, it would be zero in a linear approximation of the economy around the zero inflation steady state.

Fiscal Policy The budget constraint of the fiscal authority can be written as:

$$B_t + P_t G_t + P_t T_t = P_t \tau_t \omega_t + \frac{1}{1+i_t} B_{t+1} \tag{8}$$

where G_t denotes government purchases of the final good. Real primary surpluses are $\mathcal{S}_t$ as $\mathcal{S}_t = \tau_t \omega_t - T_t - G_t$. As we will discuss shortly, we allow the rate of labor income taxation τ_t to depend in a continuous but otherwise arbitrary fashion on the level of aggregate output in the economy: $\tau_t = \boldsymbol{\tau}(Y_t)$. We assume throughout that lump-sum transfers T_t adjust as needed to ensure fiscal solvency: fiscal policy is "*passive*" in the sense of Leeper (1991) and this is not an environment where the *fiscal theory of the price level* (FTPL) is at play. We make this assumption to highlight that in the presence of incomplete markets, fiscal policy crucially affects the effects of monetary policy even when it is 'passive'. Our results identify a new sense in which fiscal policy matters in a way which is logically distinct from the FTPL.

2.5 Market Clearing

The aggregate resource constraint implies $c_t + G_t = Y_t$ where $c_t = \int_0^1 c_t^i di$ denotes aggregate consumption.

3 Characterizing General Equilibrium

In this section we characterize equilibrium in our HANK economy. We start by solving the decision problem of each household.

3.1 Household decisions

The virtue of assuming CARA utility is that it allows us to characterize the decisions of each household in closed form. The following proposition characterizes each household's optimal decisions.

Proposition 1 (Individual decision problem)**.** *Given a sequence of real interest rates, aggregate output and idiosyncratic risk* $\{r_t, y_t, \sigma_{y,t}\}$*,*[10] *and initial wealth* a_{t-1}^i *each household's consumption decision can be expressed as:*

$$c_t^i = \mathcal{C}_t + \mu_t \left(a_t^i + y_t^i\right) \tag{9}$$

where $a_t^i = A_t^i / P_t$ *is real net worth at the start of date* t*,* μ_t *is the* marginal propensity to consume *out of cash-on-hand* $(a^i + y^i)$ *at date* t*, and* $\mathcal{C}_t$ *is the common component of consumption across households.* $\mathcal{C}_t$ *and* μ_t *solve the following recursions:*

$$\mathcal{C}_t \left[1 + \mu_{t+1} (1 + r_t)\right] = -\frac{1}{\gamma} \ln \beta (1 + r_t) + \mathcal{C}_{t+1} + \mu_{t+1} y_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \tag{10}$$

$$\mu_t = \frac{\mu_{t+1} (1 + r_t)}{1 + \mu_{t+1} (1 + r_t)} \tag{11}$$

Proof. See Appendix A. □

[10] We restrict attention to sequences of interest rates for which there exists a terminal date $T < \infty$ after which $r_t > 0$.

Equation (9) shows that individual consumption can be decomposed into an aggregate component $\mathcal{C}_t$ and an idiosyncratic component $\mu_t(a^i_t + y^i_t)$. The idiosyncratic component states that each household has the same marginal propensity to consume (MPC) μ_t out of cash-on-hand at any date t. This is a deliberate choice: to highlight the distinct roles of precautionary savings and MPC heterogeneity, we begin by abstracting altogether from the latter in order to focus on the former. We introduce MPC heterogeneity in Section 6.

Iterating equation (11) forward reveals that the MPC today depends positively on the future path of real interest rates. Consumption responds to the present value of lifetime income, not current income per se; when interest rates are high, current income is a larger fraction of this present value, so consumption responds more to current income. This can be seen clearly in the case of constant real interest rates where $\mu_t = \frac{r}{1+r}$ is constant across time and is simply the annuity value of a additional dollar of income today.[11]

The aggregate component $\mathcal{C}_t$ can be decomposed into 3 terms. To see this, solve (10) forwards to get:

$$\mathcal{C}_t = \underbrace{\sum_{s=1}^{\infty} Q_{t+s|t} \frac{\mu_t}{\gamma \mu_{t+s}} \ln \left[\frac{1}{\beta \left(1 + r_{t+s-1}\right)} \right]}_{\text{impatience}} + \underbrace{\mu_t \sum_{s=1}^{\infty} Q_{t+s|t} y_{t+s}}_{\text{PIH}} - \underbrace{\frac{\gamma \mu_t}{2} \sum_{s=1}^{\infty} Q_{t+s|t} \mu_{t+s} \sigma^2_{y,t+s}}_{\substack{\text{precautionary} \\ \text{savings}}} \tag{12}$$

where $Q_{t+s|t} = \prod_{k=0}^{s-1} \frac{1}{1+r_{t+k}}$. The first term in (12) is standard and reflects the effect of impatience and interest rates on savings: if interest rates are higher relative to β, current consumption is lower as households wish to save more. The second term reflects the permanent income hypothesis: higher expected discounted lifetime income increases current consumption.[12] The final term reflects the precautionary savings motive. To the extent that households are risk averse, $\gamma > 0$, higher future income risk lowers current consumption by increasing the desire to save. This third term indicates the effect of uninsurable income risk on aggregate consumption - if households face no idiosyncratic income risk, this term is zero and households are *permanent income consumers*. Note that what matters for the precautionary savings channel is the variance of *consumption*, not income. A given level of income risk σ^2_y depresses current consumption more when the sensitivity of consumption to income, i.e. the MPC μ_t, is higher. In our framework, this sensitivity varies over time depending on the path of real interest rates. This effect is absent in tractable HANK models which impose the zero liquidity limit (Ravn and Sterk, 2017b; Werning, 2011; McKay et al., 2017). In these models, households who are on their Euler equation anticipate that their consumption will be equal to their income in all future periods; the precautionary savings channel is present, but its strength is not affected by variations in the sensitivity of consumption to income. However, HANK models with CRRA preferences do feature this channel if the zero liquidity limit is not imposed.

3.2 Demand Block

We abstract from MPC heterogeneity across households, which permits aggregation despite a non-degenerate distribution of wealth and allows us to focus on precautionary savings. This also helps clarify the sense

[11] Caballero (1990), Weil (1993) and Wang (2003) solved a generalized version of this decision problem while assuming that the real interest rate was constant.

[12] Recall that the expected future discounted lifetime income a household is common across all households and is the same as the discounted future value of aggregate income or GDP. This follows from our assumption that individual labor endowments are i.i.d..

in which MPC heterogeneity is, or is not, necessary for heterogeneity to affect aggregate outcomes, as discussed in Section 6. The next proposition states the aggregation result formally.

Proposition 2 (Aggregation)**.** *The* aggregate consumption function *is given by:*

$$c_t = \int_0^1 c_t^i di = \mathcal{C}_t + \mu_t \left(a_t + y_t\right) \tag{13}$$

where $a_t = \int_0^1 a_t^i di$*.*

Proof. See Appendix B. □

In equilibrium, asset and goods markets clear, i.e. $c_t = Y_t$ and $a_t = \frac{B_t}{P_t}$. Appendix B shows that plugging these conditions in (13) yields the *aggregate IS equation*:

$$Y_t = Y_{t+1} - \frac{\ln \beta \left(1 + r_t\right)}{\gamma} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} + G_t - G_{t+1} \tag{14}$$

Absent risk ($\sigma_y = 0$) this *exact aggregate Euler equation* looks very much like the linearized IS equation from the 3 equation RANK model (except that this is not linearized). When $\sigma_y^2 > 0$, (14) also features a precautionary savings term which depends on 3 factors: risk aversion γ, the sensitivity of individual consumption to individual income μ_{t+1}, and idiosyncratic income risk, σ_y^2. As described above, idiosyncratic risk depends on many factors, most notably fiscal policy. More redistribution through higher taxes and transfers reduces the variance of after-tax income and diminishes the precautionary savings effect.[13]

3.3 Phillips Curve

The solution to the pricing problem of a firm described in section 2.3 is given by:

$$\Psi \Pi_t \left(\Pi_t - 1\right) = 1 - \theta \left(1 - x_t^{\frac{1-\alpha}{\alpha}}\right) + \Psi \left(\Pi_{t+1} - 1\right) \Pi_{t+1} \left[\frac{1}{1 + r_t} \frac{x_{t+1}}{x_t}\right] \tag{15}$$

where we have imposed a symmetric equilibrium. Aggregate real dividends can then be written as $\bar{D}_t = x_t - \frac{x_t^{\frac{1}{\alpha}}}{\alpha}$. As $\Psi \to \infty$, equation (15) implies $\Pi_t = 1$ for all t. We refer to this case as the "*rigid price benchmark*". In the "*flexible price benchmark*" ($\Psi = 0$), (15) implies that output is given by:

$$Y^* = \left[\frac{\theta \left(1 - \alpha\right) + \alpha}{\theta}\right] \left[\frac{\alpha \left(\theta - 1\right)}{\theta}\right]^{\frac{\alpha}{1-\alpha}} \tag{16}$$

In summary, despite being an incomplete markets model, given a path of marginal taxes $\{\tau_t\}$, the entire model can be summarized by the following equations which describe the dynamics of only aggregate

[13] Interestingly, the level of government debt does not directly enter the IS equation. Consider two economies with different levels of initial government debt. In each economy, the lifetime budget constraint of the government must hold without a bubble term, so the economy with higher initial government debt must also have a higher present discounted value of primary surpluses. Suppose this difference is entirely accounted for by a lower path of lump sum transfers, T_t in the high debt economy. Then the IS equation (14) - and in fact all equilibrium outcomes - will be identical in the two economies. In this sense, *conditional on a path of marginal taxes* τ_t, government debt is not a state variable in this economy despite incomplete markets.

variables: the IS equation (14), the MPC recursion (11), the Phillips curve (15), the definition of GDP Y_t (6), the monetary policy rule (7) and finally, the Fisher equation $\frac{1+i_t}{\Pi_{t+1}} = 1 + r_t$.

3.4 The cyclicality of income risk in General Equilibrium

So far we have not described the properties of individual income risk $\sigma^2_{y,t}$ which enters the individual household decision problem, and thus the aggregate IS equation (14). We now show that in equilibrium, $\sigma^2_{y,t}$ depends on aggregate output Y_t according to some function $\sigma^2_{y,t} = \boldsymbol{\sigma^2}(Y_t)$. $\boldsymbol{\sigma^2}(Y)$ depends endogenously on many factors - the cyclicality of wages, time varying dividend policies and in particular, fiscal policy. For our purposes, however, this rich array of factors affecting the level and cyclicality of income risk can be summarized by $\boldsymbol{\sigma^2}(Y)$, as we now explain.

Plugging in real wages (5) and the dividend function (3) into equation (1) yields the following expression for each household's income:

$$y^i_t \quad = \quad \left[\Big(1 - \boldsymbol{\tau}(Y_t)\Big)\boldsymbol{\omega}(Y_t) + \boldsymbol{\delta}(Y_t)\right]\left(\ell^i_t - 1\right) + y_t \tag{17}$$

where y_t denotes mean household income and is defined by $y_t = (1 - \boldsymbol{\tau}(Y_t))\,\boldsymbol{\omega}(Y_t) + \bar{D}_t + T_t$, where $\boldsymbol{\omega}(Y)$ defines the equilibrium real wage consistent with net output being Y.[14] Thus, individual income is normally distributed $y_{i,t} \sim N\left(y_t, \boldsymbol{\sigma^2}(Y)\right)$ and can be summarized by its mean y_t and variance:

$$\boldsymbol{\sigma^2}(Y) = \left[\Big(1 - \boldsymbol{\tau}(Y_t)\Big)\boldsymbol{\omega}(Y)^{1/\alpha} + \boldsymbol{\delta}(Y)\right]^2 \boldsymbol{\sigma^2_\ell}(Y) \tag{18}$$

Equation (18) shows that fiscal policy can affect the **level of income risk**. Trivially, by setting $\tau = 1$, the fiscal authority can totally eliminate labor income risk. In this case, the fiscal authority confiscates each household's labor income and then returns an equal share to each household, thus eliminating any variation in income arising from stochastic endowment shocks. Similarly, higher $\boldsymbol{\delta}$, i.e. more unequally distributed dividends increases income risk. Finally, a higher level of endowment/employment risk $\boldsymbol{\sigma^2_\ell}$ increases income risk to the extent that redistribution through the tax and transfer system is less than perfect ($\boldsymbol{\tau} < 1$) or dividends are unequally distributed ($\boldsymbol{\delta} > 0$).

Define the **cyclicality of income risk** as $\frac{\boldsymbol{\sigma^2}(Y)}{dY}$. This answers the following question: Supposing all exogenous variables were held fixed, and aggregate income was higher than its steady state level, would the variance of idiosyncratic income be higher or lower?[15] Equation (19) shows that the cyclicality of income risk, so defined, depends on 4 factors: (i) the cyclicality of wages $\boldsymbol{\omega}'(Y)$; (ii) the cyclicality of labor taxes, $\boldsymbol{\tau}'(Y)$; (iii) firms' dividend policy, i.e. whether dividends are more or less unequally distributed in good times compared to bad, $\boldsymbol{\delta}'(Y)$; (iv) the cyclicality of labor endowment risk $\frac{d\boldsymbol{\sigma^2_\ell}(Y)}{dY}$. This last factor can be thought of as unemployment risk which is not explicitly modeled here: if the probability of becoming unemployed is greater in recessions, i.e the probability of drawing a low labor endowment if higher when

[14] The function $\boldsymbol{\omega}(Y)$ solves $Y = \left(\frac{\alpha}{1-\alpha}\right)^\alpha \omega^\alpha - \left(\frac{\alpha}{1-\alpha}\right)\omega$ with the understanding that we only consider the smaller of the two solutions for ω. Also, note for future reference that mean household income is equal to GDP minus fiscal surplus minus government expenditures $y_t = Y_t - \mathcal{S}_t - G_t$.

[15] In a richer model featuring aggregate shocks, this notion of cyclicality - which is the relevant one when discussing determinacy and policy puzzles - need not coincide with the definition used in the empirical literature, namely the correlation between income risk and some measure of aggregate economic activity (Storesletten et al. (2004), Guvenen et al. (2014)). In our simple model without aggregate shocks, though, the definitions are essentially equivalent.

Y is low, then $\frac{d\boldsymbol{\sigma}_{\ell}^{2}(Y)}{dY} > 0$.

$$\frac{d\boldsymbol{\sigma}^{2}(Y)}{dY} = 2\boldsymbol{\sigma}(Y)\boldsymbol{\sigma}_{\ell}(Y)\left\{\underbrace{\left(1-\boldsymbol{\tau}\left(\boldsymbol{Y}\right)\right)\boldsymbol{\omega}'\left(Y\right)}_{\substack{\text{cyclicality of} \\ \text{real wages}}} - \underbrace{\boldsymbol{\tau}'\left(Y\right)\boldsymbol{\omega}\left(Y\right)}_{\substack{\text{cylicality of} \\ \text{taxes}}} + \underbrace{\boldsymbol{\delta}'\left(Y\right)}_{\substack{\text{cyclicality of} \\ \text{dividend risk}}}\right\} + \underbrace{\frac{\boldsymbol{\sigma}^{2}\left(Y\right)}{\boldsymbol{\sigma}_{\ell}^{2}\left(Y\right)}\frac{d\boldsymbol{\sigma}_{\ell}^{2}(Y)}{dY}}_{\substack{\text{cyclicality of} \\ \text{employment risk}}} \tag{19}$$

Just as the level of taxation affects the level of risk, the cyclicality of fiscal policy affects the cyclicality of risk, i.e. $\frac{\boldsymbol{\sigma}^{2}(Y)}{dY}$. For example, if labor income tax rates are lower in recessions and higher in booms ($\boldsymbol{\tau}'(Y) > 0$), this tends to make income risk countercyclical. Conversely, if the government paid lump sum transfers in recessions financed by proportional taxes, i.e. $\boldsymbol{\tau}'(Y) < 0$, this would tend to make income risk more procyclical. As we will see, the cyclicality of income risk emerges as the central factor determining whether, and how, HANK economies are different from RANK.

Since fiscal policy itself affects the cyclicality of risk, it plays a vital role in mediating the effect of monetary policy in an incomplete markets economy. This interplay of fiscal and monetary policy is logically distinct from the well known issues discussed by Leeper (1991), Woodford (1996) and Cochrane (2017b) among others concerning whether the fiscal authority raises sufficient surpluses to remain solvent along any price path. In our environment fiscal policy always ensures solvency and thus is passive. However, *how* the fiscal authority raises these surpluses, and how this varies with economic activity, affects the cyclicality of income risk in equilibrium and thus the effects of monetary policy and shocks.

Armed with these results, we start by evaluating the key factor which affects determinacy of equilibria in HANK models.

4 Determinacy of equilibrium in HANK economies

Recent work by Ravn and Sterk (2017b) argues that the Taylor principle - nominal rates should respond more than one-for one to inflation - is not sufficient to ensure determinacy in an economy with incomplete markets. Conversely, Auclert et al. (2017) numerically find that in HANK models, the Taylor Principle may not even be necessary for determinacy, provided that income risk is not too countercyclical. Our analytically tractable framework sheds light on both results. We derive a new Taylor Principle which crucially depends on the cyclicality of income risk. As we show next, procyclical income risk makes indeterminacy less likely, so a smaller Taylor rule coefficient below 1 suffices for determinacy, while countercyclical risk makes indeterminacy more likely, so a larger coefficient above 1 is required.

In order to proceed, we linearize the model around the steady state in which output is equal to its flexible price steady state level given by Y^* defined in (16).[16] Linearizing equation (14) around Y^* and

[16] It is immediate from equation (14) that the steady state of this economy must satisfy:

$$\frac{(1+r)^2 \ln\left[\frac{1}{\beta(1+r)}\right]}{\gamma^2 r^2} = \frac{1}{2}\boldsymbol{\sigma}^{2}(Y)$$

The LHS is a decreasing function of r. In other words, a high level of idiosyncratic risk must be accompanied by a lower level of real interest rates in order to clear the savings market. It is straightforward to see that this equation has a unique solution for r which is positive since the LHS is monotonically decreasing in r and asymptotes to infinity as $r \to 0$. Another advantage of our CARA economy is that we can establish uniqueness of steady states. Toda (2017) shows that in a similar economy with CARA utility, there may exist multiple steady state interest rates if the labor endowment process is not i.i.d.

$\mu = \frac{r}{1+r}$, the demand block can be written as:

$$\hat{Y}_t = \Theta \hat{Y}_{t+1} - \frac{1}{\gamma}\left(i_t - \pi_{t+1}\right) - \Lambda \hat{\mu}_{t+1} \tag{20}$$

$$\hat{\mu}_t = \tilde{\beta}\hat{\mu}_{t+1} + \tilde{\beta}\left(i_t - \pi_{t+1}\right) \tag{21}$$

where

$$\Theta = 1 - \frac{\gamma\mu^2}{2}\frac{d\boldsymbol{\sigma}^{\mathbf{2}}(Y^*)}{dY} \quad , \quad \Lambda = \gamma\mu\sigma_y^2 \quad , \quad \tilde{\beta} = \frac{1}{1+r}$$

and $\hat{Y}_t, \hat{\mu}_t$ denotes the deviation of Y_t and μ_t from their steady state values.

Equation (20) is a dynamic IS equation which relates aggregate output today to output tomorrow. In the standard RANK model, since there is no idiosyncratic risk, $\boldsymbol{\sigma} = \frac{d\boldsymbol{\sigma}(Y)}{dY} = 0$, we have $\Theta = 1$ and $\Lambda = 0$, rendering (20) identical to the standard linearized IS equation. However, the presence of idiosyncratic risk can change this. If risk is procyclical ($\frac{d\boldsymbol{\sigma}(Y)}{dY} > 0$), then $\Theta < 1$ and if risk is countercyclical $\left(\frac{d\boldsymbol{\sigma}(Y)}{dY} < 0\right)$, then $\Theta > 1$. If risk is acyclical then $\frac{d\boldsymbol{\sigma}(Y)}{dY} = 0$ and $\Theta = 1$ as in RANK.

To understand why the cylicality of risk affects Θ, consider a scenario in which the path of real interest rates are fixed and (20)-(21) can be written as:

$$\hat{Y}_t = \Theta \hat{Y}_{t+1} \tag{22}$$

and $\hat{\mu}_t = 0$, $\forall t$. Θ measures the sensitivity of output today to output tomorrow. Modestly **procyclical risk** implies that current spending moves less than one for one with future spending $\Theta \in (0, 1)$. Intuitively, suppose that at date 0, households conjecture that output at date 1 is going to be higher than steady state, i.e. $\hat{Y}_1 > 0$. This belief about higher output at date 1 affects aggregate demand at date 0 in two ways. The first is associated with the permanent income channel. Anticipating higher income at date 1, agents demand more consumption at date 0. This increased date 0 demand raises income at date 0, further raising consumption at date 0. Overall, via the permanent income channel, an increase in $\hat{Y}_1$ would increase $\hat{Y}_0$ one-for-one. However, there is also a second effect. Since income risk is procyclical, higher output at date 1 also increases the idiosyncratic risk agents face at date 1. This tends to reduce date 0 consumption, and thus output, via the precautionary savings channel. Overall, an increase in $\hat{Y}_1$ tends to increase date 0 output less than one-for-one.

Equation (22) makes it clear that procyclical income risk delivers a "*discounted*" Euler equation as in McKay et al. (2015, 2017). Our derivation of a similar result clarifies that procyclical income risk - rather than market incompleteness per se - is responsible for generating a discounted Euler equation. Indeed, the model in McKay et al. (2017) features strongly procyclical income risk: "low productivity households receive a constant transfer from the government while high productivity households receive all cyclical wages and dividends, minus the acyclical transfers." As a result, the income gap between high and low productivity households (equivalently, the variance of individual income) is highest in booms and lowest in recessions.[17] Countercyclical risk instead delivers an "*explosive*" rather than a discounted Euler equation as we now show.

[17] In a similar vein Werning (2015) argued that procyclical risk and not incomplete markets per se accounts for McKay et al. (2015)'s resolution of the Forward guidance puzzle. We clarify the difference between our work and Werning (2015) in Section 5.1.

Countercyclical risk instead implies that that current spending moves more than one for one with future spending $\Theta > 1$. Suppose that at date 0, households contemplate a lower output than steady state at date 1, $\hat{Y}_1 < 0$. This directly depresses consumption via the permanent income channel as agents expect to be poorer in the future; on its own, this would tend to make date 0 output fall one-for-one with date 1 output. In addition, however, agents understand that lower date 1 output implies higher idiosyncratic risk at date 1. This further lowers consumption demand at date 0 via the precautionary savings channel. Overall, $\hat{Y}_0$ falls more than one-for-one with $\hat{Y}_1$. Countercyclical unemployment can naturally generate countercyclical income risk, as discussed by Challe and Ragot (2016), Challe et al. (2017), Ravn and Sterk (2017a) and Ravn and Sterk (2017b) among others. Lower expected future output depresses aggregate demand today, not just because agents feel poorer, but also because they face a higher risk of becoming unemployed. While our setup does not feature unemployment, one can think of countercyclical income risk as an increase in the probability of drawing low labor endowments (i.e. an increase in the variance of labor endowment) in recessions.

Finally, **acyclical risk** implies that current spending moves one-for-one with future spending ($\Theta = 1$) as in RANK. While agents face risk in the acyclical case, changes in the perception about GDP in the future do not affect their perceptions of this risk and thus do not affect current spending via the precautionary savings channel.

Even though acyclical risk implies $\Theta = 1$, however, the HANK economy still differs from the RANK economy in this case owing to the third term on the RHS of (20). This term arises from interactions between the precautionary savings motive and variations in real interest rates. Recall that $\hat{\mu}_t$ denotes the log-deviation of households' (common) MPC. When the MPC is high, individual consumption is more responsive to individual income, and so a given level of income risk translates into more volatile consumption, and thus a stronger precautionary savings motive. Thus when $\hat{\mu}_{t+1}$ is high, households seek to reduce consumption today relative to tomorrow. $\hat{\mu}_{t+1}$ in turn depends on the whole future path of interest rates as shown in (21). In this sense, the $-\Lambda\hat{\mu}_{t+1}$ term in (20) and equation (21) represent a novel channel of monetary policy: tighter monetary policy increases the sensitivity of individual consumption to individual income shocks, raising consumption risk and reducing demand via the *precautionary savings* channel. This is in addition to the standard *intertemporal substitution* channel of monetary policy, represented by the second term in equation (20). If $\boldsymbol{\sigma}(Y) = 0$, $\Theta = 1$, $\Lambda = 0$, and this economy reduces to the standard 3-equation RANK model.

Finally, as in the canonical RANK model, linearizing the Phillips curve and Taylor rule around the zero inflation steady state yields:[18]

$$\hat{\pi}_t = \tilde{\beta}\pi_{t+1} + \kappa\hat{Y}_t \tag{23}$$

$$i_t = \Phi_\pi \pi_t \tag{24}$$

Equation (23) is the standard linearized Phillips curve and (24) denotes the interest rate rule where i_t denotes the log deviation of $1 + i_t$ from steady state while, π_t denotes the log deviation of inflation Π_t from steady state $\Pi = 1$. Altogether, this gives us a 4 equation model, which nests the RANK economy as a special case. The difference between HANK and RANK is concentrated in the aggregate demand block, represented by the first two equations.

[18] Appendix C derives the linearized model.

Before characterizing determinacy, it is useful to make the following assumption.

Assumption 1. *Income risk is not too countercyclical:* $\Theta \in (0, \bar{\Theta})$ *where* $\bar{\Theta}$ *is defined in Appendix D and is greater than 1 for* σ_y^2 *sufficiently small.*

The following Proposition describes local determinacy in this economy.

Proposition 3 (An income-risk augmented Taylor Principle)**.** *The following condition on the interest rate rule (24) is necessary and sufficient for equilibrium to be locally determinate:*

$$\Phi_\pi > 1 + \frac{\gamma}{\kappa}\left[\frac{\left(1-\tilde{\beta}\right)^2}{\left(1-\tilde{\beta}\right)+\gamma\tilde{\beta}\Lambda}\right]\underbrace{(\Theta-1)}_{\substack{\text{cyclicality}\\ \text{of}\\ \text{income}\\ \text{risk}}} \tag{25}$$

Proof. See Appendix D. □

If income risk is acyclical, $\Theta = 1$, then determinacy requires $\Phi_\pi > 1$, as in the RANK model. Thus the introduction of incomplete markets does not necessarily change the determinacy properties of equilibrium. Away from the acyclical risk benchmark, procyclical income risk tends to make determinacy more likely, while countercyclical risk makes it less likely, as we found in the fixed price limit. More precisely, if income risk is procyclical ($\Theta < 1$), determinacy obtains even if the standard Taylor principle is violated and $\Phi_\pi < 1$ - unlike in the 3-equation RANK model, raising nominal rates more than one for one with inflation is not *necessary* to ensure determinacy. Indeed, if risk is sufficiently procyclical, then determinacy is guaranteed even under a nominal interest rate peg, contrary to the classic result of Sargent and Wallace (1975). A HANK economy with procyclical income risk contains a powerful additional stabilizing force: a higher path of output implies higher risk, which reduces demand and prevents the rise in output from being self-fulfilling.[19] Our results are consistent with Auclert et al. (2017) who find numerically that in a HANK economy with CRRA preferences, determinacy obtains with a lower Taylor rule coefficient, or under a real interest rate peg, when income risk is sufficiently procyclical.

Conversely, if income risk is countercyclical ($\Theta > 1$), the standard Taylor principle $\Phi_\pi > 1$ is not even *sufficient* for determinacy, unlike in the RANK model. Countercyclical risk creates an additional destabilizing force: lower output implies higher risk, reducing demand and allowing the fall in output to become self-fulfilling. Monetary policy must respond more aggressively to prevent such self-fulfilling fluctuations. This result resonates with the findings of Ravn and Sterk (2017b) who argue that even if monetary policy satisfies the Taylor principle, equilibrium in a HANK model may be indeterminate.[20] Our analysis makes clear that this result is not a property of HANK models in general - indeed we have just seen that procyclical income risk makes indeterminacy less likely in such models. Instead, indeterminacy is more likely in HANK models only when income risk is countercyclical, as in Ravn and Sterk (2017b).

Finally, all else equal, a higher Λ weakens the extent to which pro- or counter-cyclical income risk warrants a deviation from the classic Taylor principle. A higher Λ makes monetary policy more powerful:

[19] This argument hinges on the fact that output is partially demand-determined in this economy with nominal rigidities. In the flexible price limit, $\kappa \to \infty$, we recover the standard Taylor principle $\Phi_\pi > 1$, whatever the cyclicality of income risk.

[20] Auclert et al. (2017) also numerically find indeterminacy in the case of sufficiently countercyclical income risk under a real interest rate peg or when the coefficient on inflation is not sufficiently above 1. We thank an *anonymous* referee for pointing this out.

smaller changes in interest rates have a larger effect on aggregate output, operating not just through the intertemporal channel but also by changing the passthrough from income to consumption shocks and affecting desired precautionary savings.[21]

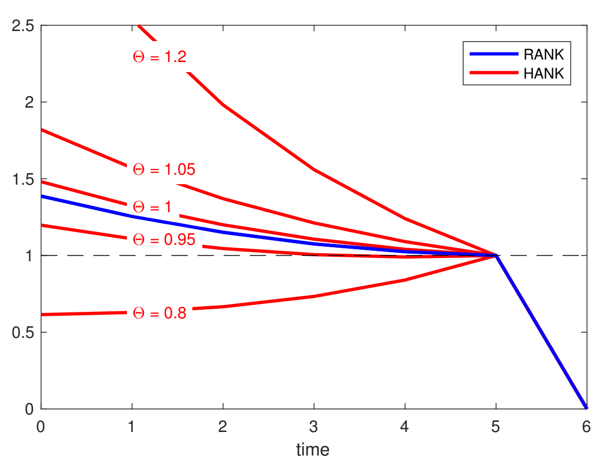
5 Some RANK policy puzzles

Recent work has argued that RANK models make unrealistic predictions about the depth of recessions and deflation during liquidity trap episodes, the size of government spending multipliers at the ZLB, and the effects of forward guidance. These perceived shortcomings have often been explained by the notion that the intertemporal substitution channel is 'too strong' in the RANK model, in which the representative agent is essentially a permanent income consumer, and households are on their Euler equation at all point in time. This diagnosis suggests that moving towards a HANK model might reverse the three 'unrealistic' predictions described above. Our analytically tractable framework allows us to shed some light on how, if at all, market incompleteness might actually affect these predictions.

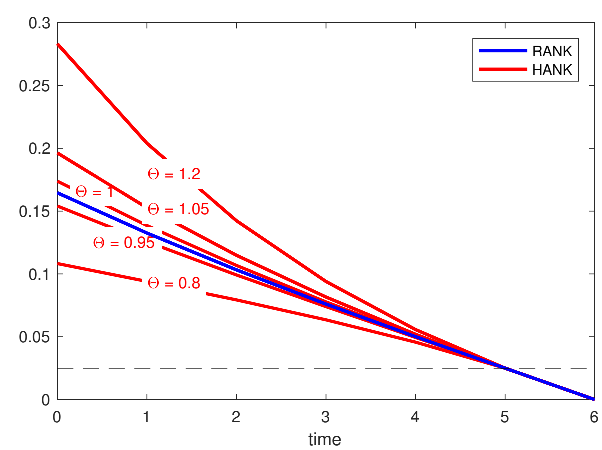
5.1 Forward guidance

As in Del Negro et al. (2015) and McKay et al. (2015), the effect of forward guidance in the RANK model is best illustrated with a simple experiment. Suppose the monetary authority announces at date t a temporary decline in the short-term nominal interest rate at date $t+k$: $i_{t+k} = -\varepsilon < 0$, $i_{t+s} = 0$ for all $s \neq k$. How does the effect of this shock on date t output depend on the horizon of forward guidance k? In the RANK version of our economy $\Theta = 1$, $\Lambda = 0$, and so iterating the IS equation forward yields

$$\hat{Y}_t = -\gamma^{-1} \sum_{k=0}^{\infty} (i_{t+k} - \pi_{t+k+1})$$



(a) Output



(b) Inflation

Figure 1. Response of output and inflation to a unit drop in nominal interest rates 5 periods in the future. Blue line indicates RANK economy; red lines indicate HANK economies with lower lines corresponding to lower values of Θ.

[21] As mentioned elsewhere, this channel would be absent if we had considered an economy with zero liquidity, in which case the passthrough from income to consumption shocks is always equal to 1 and $\hat{\mu}_t = 0$.

Under rigid prices ($\kappa = 0$), $\pi_{t+k} = 0$, and nominal and real rates move by the same amount. In this case, whatever the horizon of forward guidance k, output and consumption increase by ε at date t and remain at this level until $t + k$ - announcements about far future interest rates are *equally* as effective as contemporaneous changes in interest rates. Under sticky (not rigid) prices ($\kappa > 0$), announcements about far future interest rates are even *more* effective than contemporaneous changes. Inflation can also be written as the present discounted value of future output gaps:

$$\pi_t = \kappa \sum_{k=0}^{\infty} \tilde{\beta}^k \hat{Y}_{t+k}$$

A larger k (more distant changes in policy) implies output will be high for longer, creating a larger increase in inflation, which in turn reduces real interest rates and stimulates output further. In RANK, all these effects can be understood in terms of the *intertemporal* channel of monetary policy. Lower real interest rates - caused both by a commitment to lower nominal rates and the resulting higher expected inflation - induce a declining path of consumption. With date $t+k$ output fixed at Y^* (implicitly by an active Taylor rule after date $t+k$), lower consumption *growth* implies a higher *level* of consumption, and output, today.

The blue line in Figure 1a shows the response of output to the announcement of a unit cut in nominal interest rates at date $t + 5$.[22] The response of output remains positive until the announced change in policy is enacted at date 5, with the largest effect on the day of announcement - date 0. Figure 1b shows that inflation behaves similarly, jumping at date 0 in anticipation of a sustained period of higher output and then gradually declining. Figure 2 shows how the impact effect of a future one-time cut in interest rates depends on the horizon of the policy change. Again the blue curve describes outcomes in RANK. Announced future policy changes are more effective than contemporaneous policy changes. Del Negro et al. (2015) dub this phenomenon the *forward guidance puzzle* (FGP) .

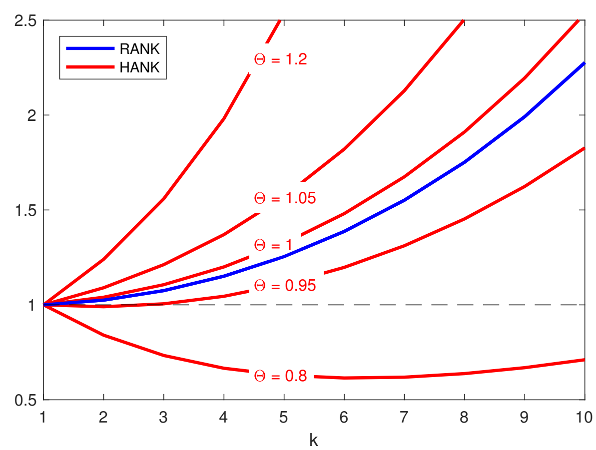


Figure 2. Impact change in output as a function of horizon of forward guidance k. Monetary policy cuts nominal interest rates only at date $t + k$

Now consider the same experiment in our HANK economy. Again, start with the case of rigid prices.

[22] Henceforth, whenever we plot graphs, we will use the following parameters. We set the coefficient of absolute risk aversion $\gamma = 1$, discount factor $\beta = 0.98$, the slope of the Phillips curve $\kappa = 0.025$. In addition, we set the steady state level of idiosyncratic risk to $\sigma_y^2 = 100$. While this does not matter in the RANK economy, this level of steady state risk generates a steady state real interest rate of $r = \tilde{\beta}^{-1} - 1 = 0.0126$ in the HANK economy. This exercise is not quantitative in nature and is only for illustration purposes.

Iterating the IS equation forward now yields

$$\hat{Y}_t \;=\; -\gamma^{-1}\sum_{k=0}^{\infty}\Theta^k i_{t+k} - \Lambda\sum_{k=0}^{\infty}\Theta^k\sum_{s=1}^{\infty}\tilde{\beta}^s i_{t+k+s} \tag{26}$$

where we have used the fact that $\hat{\mu}_{t+k} = \sum_{s=0}^{\infty}\tilde{\beta}^{s+1} i_{t+k+s}$ with fixed prices. Consequently, one can express the sensitivity of output at date t to interest rate changes at date $t+k$ as:[23]

$$\frac{d\hat{Y}_t}{di_{t+k}} = -\gamma^{-1}\Theta^k - \Lambda\sum_{s=1}^{k}\tilde{\beta}^s\Theta^{k-s} \tag{27}$$

Suppose first that idiosyncratic income risk is countercyclical, so $\Theta > 1$. With $\Theta > 1$, Θ^k is increasing in k and so is $\sum_{k=1}^{k}\tilde{\beta}^s\Theta^{k-s}$. Thus, announcements in the far future are *more* effective in stimulating demand than contemporaneous changes in policy even with fixed prices:

$$\left|\frac{d\hat{Y}_t}{di_{t+k+1}}\right| > \left|\frac{d\hat{Y}_t}{di_{t+k}}\right| \;,\; \forall k \geq 0$$

The *intertemporal channel* is operative in both the HANK and RANK economies. Lower future interest rates induce lower consumption growth and higher consumption (and output) on impact, in both economies. However, the FGP is more severe in this HANK economy than in RANK for two new reasons associated with the *precautionary savings channel.* First, higher future output reduces idiosyncratic income risk. Anticipating higher income and hence less risk at date $t+k$, households at date $t+k-1$ increase spending more than one for one, leading to a larger boom at date $t+k-1$. This in turn leads to an even larger boom at date $t+k-2$, and so forth. Second, lower real interest rates make consumption less responsive to current income, i.e $\hat{\mu} < 0$. Thus, for a given path of income risk, consumption risk - which ultimately matters for precautionary savings - is lower, further boosting spending today. Overall, incomplete markets can substantially amplify the FGP relative to RANK if income risk is countercyclical.

If income risk was acyclical $\Theta = 1$, while the first effect just described would be absent, the second effect would still be operative. Indeed with $\Theta = 1$, we can express $\left|\frac{d\hat{Y}_t}{di_{t+k}}\right|$ as:

$$\left|\frac{d\hat{Y}_t}{di_{t+k}}\right| = \gamma^{-1} + \beta\Lambda\frac{1-\tilde{\beta}^k}{1-\tilde{\beta}}$$

which is strictly increasing in k. While forward guidance does not affect income risk in this case (since risk does not depend on the level of aggregate economic activity) it does reduce the sensitivity of consumption to income by lowering μ_t, and thus boosts consumption by lowering desired precautionary savings.

Thus procyclical income risk ($\Theta < 1$) is essential if incomplete markets are to resolve the FGP. With $\Theta < 1$, the response of date t output to a unit reduction in interest rates at $t+k$ has two components, as can be seen from (27). The first term $\gamma^{-1}\Theta^k$ is decreasing in k, which tends to make announcements about future interest rate cuts *less* effective in stimulating demand than contemporaneous cuts. This is not

[23] In the case where $\tilde{\beta} \neq \Theta$, this can be written as $\frac{d\hat{Y}_t}{di_{t+k}} = -\gamma^{-1}\Theta^k - \beta\Lambda\frac{\Theta^k-\tilde{\beta}^k}{\Theta-\tilde{\beta}}$. In the knife-edge case where $\tilde{\beta} = \Theta$ the formula simply becomes $\frac{d\hat{Y}_t}{di_{t+k}} = -\left(\Lambda k + \gamma^{-1}\right)\Theta^k$.

because households are not forward-looking, or because they anticipate being borrowing constrained in the future; our households are infinitely lived and unconstrained. Instead, it is because idiosyncratic income risk is procyclical. At date $t+k-1$, households anticipate that the cut in the policy rate at date $t+k$ will increase output and average income, but they also expect this to generate an increase in idiosyncratic income risk. Consequently, while higher average income would induce them to increase consumption one for one, the increase in risk tends to reduce their consumption response, so spending at date $t+k-1$ increases less than one for one. By the same logic, at date $t+k-2$, households increase spending less than one for one in response to a smaller expected increase in income at date $t+k-1$, and so forth.

Even if an announced cut in future interest rates increases idiosyncratic *income* risk, it also reduces the sensitivity of individual consumption to income (μ). Thus, the overall effect on *consumption* risk, and precautionary saving, is ambiguous. For mildly procylical income risk (Θ close to 1), this second channel may still reduce desired precautionary savings on net, increasing the impact of far future interest rate cuts, and leaving the FGP unresolved. However, for sufficiently procyclical income risk ($\Theta << 1$) desired precautionary savings are increased on net, reducing the effectiveness of far future interest rate cuts.

The cyclicality of income risk remains key even when prices are not perfectly rigid ($\kappa > 0$). In this case, the FGP is more pronounced even in RANK due to the expected inflation channel; this carries over to HANK economies. Thus, even with moderately procyclical income risk, the puzzle may persist in the sense that announced changes have a larger effect on output than contemporaneous changes (See the case with $\Theta = 0.95$ in Figure 2). Nonetheless, more procyclical income risk (lower Θ) still reduces both the size of the output response to announced policy changes, and the gains (if any) from future announcements. The red lines in Figure 1a illustrate the paths of output in response to a unit cut in nominal interest rates at date 5, for various values of Θ. As can be seen, lower Θ reduces the effect of forward guidance on output at all horizons while higher Θ increases the effect; Figure 1b shows that the same is true for inflation.

McKay et al. (2015) argue that market incompleteness softens this prediction, generating behavior which can be approximated in terms of a 'discounted Euler equation', in which output today moves less than one for one with output in the far future. Werning (2015) instead argues that market incompleteness need not change the sensitivity of aggregate consumption to current and future interest rates. He presents examples with countercyclical risk in which consumption is *more* sensitive to current and (especially) future interest rates than in RANK, and argues that McKay et al. (2015)'s results might be driven by their assumption of procyclical risk. Our framework clarifies that procyclical risk is responsible for reducing the sensitivity of consumption to *future* interest rates, generating discounting in the Euler equation and weakening the FGP. However, it need not affect the sensitivity of consumption to *current* interest rates. In section 6 we show that the introduction of MPC heterogeneity (in the form of some hand to mouth agents), while it does not generate a discounted Euler equation, can reduce or increase the sensitivity of consumption to current interest rates (which remains equal to γ^{-1} in our benchmark without MPC heterogeneity). As we explain there, the discrepancy arises because our framework allows us to distinguish the cyclicality of the income of hand to mouth agents and the cyclicality of risk, while his examples do not make this distinction.

5.2 Fiscal multipliers

The textbook 3-equation RANK model predicts large declines in output and inflation during a liquidity trap, when the natural rate of interest is negative and the nominal rate is constrained by the zero lower bound (ZLB). Temporary increases in government spending during the liquidity trap have unusually large multipliers (substantially greater than 1) which grow with the duration of the liquidity trap. We now explore whether, and how, these predictions are modified in an incomplete markets model.

With government purchases and time varying β_t, our linearized Euler equation becomes:[24]

$$\hat{Y}_t = \Theta \hat{Y}_{t+1} - \gamma^{-1} (i_t - \pi_{t+1} - \rho_t) - \Lambda \hat{\mu}_{t+1} + \hat{G}_t - \hat{G}_{t+1} \tag{28}$$

where $-\rho_t$ denotes the log-deviation of β_t. We consider a scenario in which $\rho_t = -\bar{\rho} < 0$ for $t < T$ and $\rho_t = 0$ for $t > T$. Monetary policy is assumed to be constrained by the ZLB until date T: in log-deviations, $i_t = -\iota < 0$ for $t < T$. Starting at date T, we assume that monetary policy implements the zero-output gap, zero inflation equilibrium (for example, with an appropriately specified active Taylor rule) so that $i_t = 0$. Furthermore, consider a fiscal policy that sets $\hat{G}_t = g > 0$ for the duration of the liquidity trap ($0 \le t < T$) and zero thereafter. Again, it is instructive to start with the fixed price limit. Then, the fiscal multiplier can be expressed as:

$$\frac{\partial \hat{Y}_t}{\partial g} = \begin{cases} \Theta^{T-t-1} & \text{for } 0 \le t < T \\ 0 & \text{else} \end{cases} \tag{29}$$

In RANK, we have $\Theta = 1$, and (29) implies that the multiplier at each date during the liquidity trap is 1, independent of the duration of the trap T.[25] Incomplete markets, per se, need not change this: acyclical risk also implies $\Theta = 1$ and so the multiplier is 1 at all horizons.

Pro-cyclical income risk ($\Theta < 1$) reduces the multiplier below 1. Further, in this case, the multiplier is *decreasing* in the duration of the trap $T - t$; equivalently, the multiplier becomes larger as the end of the trap (and end of the increased spending) approaches. Intuitively, when households anticipate higher government spending throughout the duration of the trap, they also expect higher aggregate income; but because idiosyncratic risk is procyclical, this carries with it a higher level of risk faced by each household, inducing them to spend less when real interest rates are fixed.

Countercyclical risk ($\Theta > 1$), on the other hand, delivers a multiplier greater than 1 at all horizons and increasing in the duration of the trap. These large and increasing multipliers are not due to the expected inflation channel (Woodford, 2011; Eggertsson, 2011) which is absent since prices are assumed to be fixed. Instead, they are due to the precautionary savings channel: higher future government spending increases aggregate output which reduces idosyncratic risk. Anticipating this, households consume more for a fixed real interest rate.

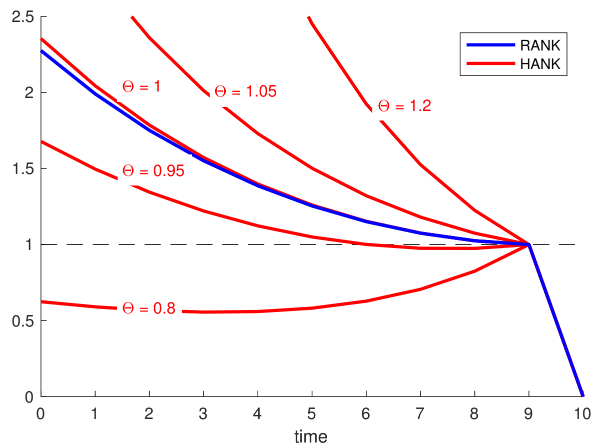
The intuition broadly carries over to an environment with sticky rather than fixed prices where the expected inflation channel is also at work. This tends to raise fiscal multipliers, especially in more protracted liquidity traps. However, it remains true that procyclical risk tends to dampen the fiscal multiplier and

[24] See Appendix B for a derivation.

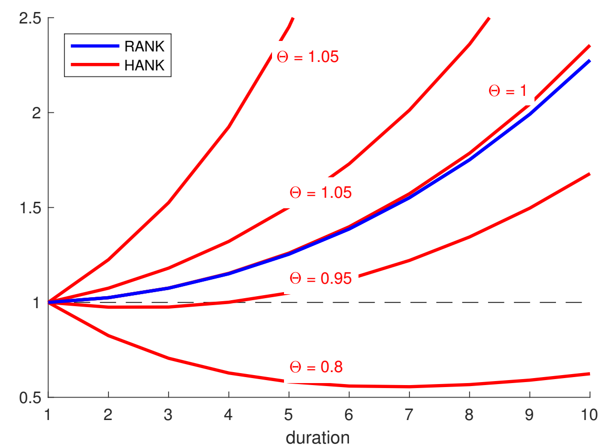
[25] We show below that with sticky rather than rigid prices, the fiscal multiplier is larger than 1 and increasing in the trap duration in RANK because of the expected inflation channel, which is missing if prices are rigid.

its dependence on the duration of the liquidity trap episode, while countercylical risk amplifies the fiscal multipliers further relative to RANK. Figure 3a plots the multiplier $\frac{d\hat{Y}}{dg}$ for a liquidity trap that lasts 10 periods (with sticky rather than rigid prices). As described above, we consider a policy which increases government purchases for the duration of the liquidity trap by some constant amount $g > 0$.

With fixed prices, multipliers were identical in RANK and the acyclical risk HANK economy. This is not literally true when prices are somewhat flexible. In RANK, higher future government spending stimulates output which raises future inflation via the Phillips curve. With nominal rates constrained by the ZLB, higher expected inflation reduces real rates, raising consumption (and output). This expected inflation channel is not just present, but is qualitatively strengthened, in the HANK economy. Lower real rates lower μ_t, the sensitivity of household consumption to income shocks. Even with income volatility fixed (acyclical risk), this reduces the volatility of future household consumption, further stimulating demand via the precautionary savings channel. In Figure 3a this second effect is quantitatively small with the blue line (RANK) lying almost on top of the red line (HANK economy with $\Theta = 1$).[26] Instead, as in the fixed price case, the most important way in which incomplete markets affect the multiplier is again, via the cyclicality of income risk. As can be seen in Figure 3a, more procyclical risk (lower Θ) lowers the multiplier while countercyclical risk amplifies the multiplier relative to RANK.



(a) Fiscal Multipliers $\frac{d\hat{Y}_t}{dg}$ given a 10 period liquidity trap.



(b) $\frac{d\hat{Y}_0}{dg}$ as a function of liquidity trap duration

Figure 3. Fiscal Multipliers

Figure 3b illustrates how the duration of the liquidity trap affects the impact government spending multiplier $d\hat{Y}_0/dg$. In RANK, the multiplier is greater than 1 and increasing in the duration of the trap. Whereas the fixed price RANK multiplier was independent of duration, here the multiplier is larger in a longer trap because of the expected inflation channel. Procyclical risk ($\Theta < 1$) reduces the impact multiplier. In fact, when income risk is sufficiently procyclical, the multiplier is decreasing in the duration of the trap. The effect of the expected inflation channel is counteracted by procyclical risk. Higher government spending raises output; the longer the duration of the spending episode, the longer output remains high, and the larger the increase in inflation (the sum of net discounted future output).[27] At the same time, if risk is procyclical, a longer episode of higher output raises risk more and thus depresses

[26] Since the effect of changes in future real interest rates on the MPC would also be present in a Bewley-Aiyagari-Huggett with CRRA preferences, this effect could conceivably be larger in that setting. We leave this issue for future work.

[27] This is the expected inflation channel.

private spending more, ceteris paribus (see equation (12)). Thus, procyclical risk can overwhelm the expected inflation channel lowering impact multipliers. If instead, risk is countercyclical, the expected inflation and precautionary savings channels work in the same direction, making the multiplier larger and strongly increasing in the duration of the liquidity trap.

6 Accounting for MPC Heterogeneity

The nascent HANK literature has emphasized the importance of MPC heterogeneity. Indeed, it is sometime suggested that incomplete markets matter to the extent that some households are or expect to be 'off their Euler equation', because this makes households less forward-looking, increases the MPC and weakens the strength of the intertemporal channel. In our model, in equilibrium, households are never off their Euler equation and there is no MPC heterogeneity. Yet the predictions of the model can differ substantially from that of the RANK model, due to the precautionary savings motive. This should not really be surprising in light of the earlier consumption literature. As is well known, in partial equilibrium, precautionary savings arises even for an unconstrained individual if the third derivative of the period utility function is positive, as it is in our CARA economy (Leland, 1968; Sandmo, 1970).[28] This precautionary saving motive can change outcomes in a NK economy, even if there is no MPC heterogeneity.

While we have deliberately abstracted from MPC heterogeneity thus far, it is straightforward to introduce this feature into our environment. We do so by adding a fraction of *hand-to-mouth* agents in the economy. Suppose now that out of the unit mass of households, only those indexed by $i \geq \eta$ are unconstrained and those with $i \in [0, \eta]$ are hand-to-mouth consumers who consume their entire after-tax incomes. With the introduction of these new agents, the economy now features MPC heterogeneity as a fraction η of agents have MPC 1 and a fraction $1-\eta$ have MPC $\mu_t < 1$. Varying η allows us to vary the average MPC in the economy which is given by $\eta + (1-\eta)\mu_t$.

As Bilbiie (2008, 2017a,b) has shown, the effect of introducing hand-to-mouth (HTM henceforth) agents into the economy depends critically on the sensitivity of these agents' income to aggregate income. Following Bilbiie (2017a), we assume that the after-tax income of HTM agents is proportional to aggregate after tax income, y_t. In particular, we assume that the average income of HTM agents, $\frac{1}{\eta}\int_0^\eta y_t^i di = \chi y_t$ where y_t denotes after-tax aggregate income.[29] Meanwhile for unconstrained households, the income process is now given by $y_t^i \sim N\left(\frac{1-\eta\chi}{1-\eta} y_t, \sigma_{y,t}^2\right)$. Constrained and unconstrained agents might have different cyclical sensitivities of income for many reasons. The simplest case in which $\chi > 1$ is that the government directly imposes lump sum taxes (proportional to GDP) on the unconstrained households and rebates the proceeds lump sum to constrained households. In other words, if the transfer to constrained agents is $(\chi - 1)Y_t$ and the tax on unconstrained agents is $\eta(\chi - 1)Y_t/(1-\eta)$, the after-tax income of both agents is as described. Similarly, $\chi < 1$ if transfers are in the other direction.

[28] See (Carroll and Kimball, 2001) for a thorough discussion of the relation between precautionary savings and liquidity in an individual decision problem. In continuous time, Achdou et al. (2017) show that even though all agents are on their Euler equation, the presence of a borrowing constraint generates precautionary savings motive even if the third derivative of period utility functions is not positive.

[29] Because these households do not make savings decisions, the distribution of incomes among these households does not affect aggregate outcomes.

In this case, Appendix E shows that the aggregate Euler equation can be written as:[30]

$$Y_t = -\frac{\Xi}{\gamma}\ln\beta\left(1+r_t\right) + Y_{t+1} - \Xi\frac{\gamma\mu_{t+1}^2}{2}\sigma_{y,t+1}^2 \tag{30}$$

where $\Xi = \frac{1-\eta}{1-\eta\chi} > 0$.[31] In order to understand the role played by χ, note that the resource constraint implies:[32]

$$Y_t = C_t = \eta\chi Y_t + (1-\eta)c_t^u \qquad \Rightarrow \qquad Y_t = \frac{1-\eta}{1-\eta\chi}c_t^u \equiv \Xi c_t^u$$

In other words, Ξ is the amount by which GDP goes up if the per capita consumption of unconstrained households goes up by 1 unit, which may be due to a cut in interest rates, lower risk which reduces precautionary savings etc. Suppose first that $\chi = 1$, i.e., constrained and unconstrained individuals have the same average income. A unit increase in the consumption of the unconstrained, directly increases GDP by $1-\eta$ units. This increase in aggregate income in turn increases the income of HTM agents by $1-\eta$; these individuals spend the whole increment, increasing aggregate consumption and further increasing aggregate GDP by $\eta(1-\eta)$. This in turn induces a third round effect and so on. The net effect is that a unit increase in c_t^u increases GDP by $(1-\eta)(1+\eta+\eta^2+\cdots) = \frac{1-\eta}{1-\eta} = 1$, however small the fraction of unconstrained individuals $1-\eta$. In the language of Kaplan et al. (2016), a larger fraction of HTM agents, η reduces the *direct* effect of an increase in c_t^u on GDP but is exactly compensated by larger *indirect* effects from the second round effects of increased spending by HTM agents. Thus, with $\chi = 1$, c_t^u moves one-for-one with Y_t, for any value of $\eta \in [0,1]$ and the aggregate Euler equation is the same as if there were no HTM agents in the economy. Notice that if $\chi = 1$, then $\Xi = 1$ and (30) is the same as equation (14)(with G_t set to 0).

However, as Bilbiie (2008, 2017a,b) has forcefully argued, this irrelevance result is special to the case where $\chi = 1$. If instead, $\chi < 1$ (the income of HTM's is less cyclically sensitive than that of unconstrained agents), then $\Xi < 1$, and an increase in the consumption of the unconstrained increases GDP less that one-for-one, especially to the extent that the fraction of HTM agents η is large. Intuitively, if the income and spending of HTM agents increases less than one-for-one with GDP, the second round multiplier effects described in the previous paragraph are naturally dampened. If instead $\chi > 1$ and HTM agents have more cyclically sensitive incomes, then $\Xi > 1$, the an increase in the consumption of unconstrained raises GDP more than one-for-one.

6.1 Interest rate sensitivity of aggregate spending

With these results in hand, it is straightforward to see how MPC heterogeneity can affect the interest rate sensitivity of aggregate spending. Since real interest rates do not directly affect the behavior of HTM agents, they affect GDP directly only via the behavior of unconstrained agents. The sensitivity of consumption growth to real interest rates for unconstrained agents is not affected by the introduction of HTM agents and equals γ^{-1} as before. But the sensitivity of GDP to the consumption of the unconstrained is given by Ξ as we described above. Thus, the overall sensitivity of GDP to real interest rates is given by $\frac{dY_t}{d\ln(1+r_t)} = \frac{dY_t}{dc_t^u}\frac{dc_t^u}{d\ln(1+r_t)} = \frac{\Xi}{\gamma} \equiv \widetilde{\gamma}^{-1}$, as shown in (30). When $\chi = 1$, MPC heterogeneity does not affect

[30] Here without loss of generality we set $G_t = T_t = \mathcal{S}_t = 0$. Appendix E contains the general specification.

[31] We focus on the economically meaningful case in which $\eta\chi < 1$, i.e. HTM agents do not receive all of the after-tax aggregate income.

[32] Setting $G_t = T_t = \mathcal{S}_t = 0$ without loss of generality

aggregate interest rate sensitivity. When $\chi < 1$, HTM agents dampen the interest rate sensitivity, while when $\chi > 1$, they amplify it.

Note that MPC heterogeneity changes the predictions of the NK model in a qualitatively different way than uninsurable idiosyncratic risk, which was the main focus of the paper up till now. MPC heterogeneity can change the contemporaneous response of GDP to interest rates, depending on the cyclicality of HTM income. While precautionary savings and uninsurable risk affect other facets of the mode, they do not affect the contemporaneous interest rate sensitivity, regardless of the cyclicality of income risk.

MPC heterogeneity and uninsurable risk are two logically distinct features (but not necessarily unrelated) of an incomplete markets economy. Our framework allows us to make this distinction explicit: taking σ_y to zero in the limit while introducing an $\eta > 0$ fraction of HTM agents gives us MPC heterogeneity without risk, while $\sigma_y > 0$ and $\eta = 0$ gives us risk but no MPC heterogeneity. More subtly, the *cyclicality* of risk is distinct from the cyclicality of HTM agents' income. Cyclicality of risk concerns the question "does risk go up or down in a recession?". A separate question is, "Who loses disproportionately from a recession?" - in particular, does income fall more or less for constrained (high MPC) agents or unconstrained agents?

Our framework clearly distinguishes between the cyclicality of risk which is given by $\frac{d\boldsymbol{\sigma}}{dY}$ and the cyclicality of HTM income which is given by χ. Recent work by Bilbiie (2017a) and Werning (2015) have emphasized the importance of both these forces. However, the analytical framework used in these papers does not permit the two factors to be distinguished - the same parameters which which generate countercyclical risk also generate also imply a higher sensitivity of HTM income. This leads Werning (2015) for example to conclude that countercyclical income risk increases the sensitivity of consumption to contemporaneous changes in interest rates.

To see this, consider the following simple example based on Werning (2015). There is a continuum of agents in the economy and two income states $y^i \in \{y_h(Y_t), y_l(Y_t)\}$ which potentially depend on aggregate income Y_t in different ways: $y_h'(Y) \neq y_l'(Y)$. The probability of switching from income state s to s' is denoted by $p_{s'|s}$. Agents can save in a zero net-supply risk free bond and cannot borrow. It is straightforward to see that in equilibrium all households consume their entire income and high income households are unconstrained, thus pricing the bond. The low income individuals are effectively HTM agents. The high income agents' Euler equation is:

$$u'\left(y_h(Y_t)\right) = \beta(1+r_t)\left[p_{l|h}u'\left(y_l(Y_{t+1})\right) + p_{h|h}u'\left(y_h(Y_{t+1})\right)\right]$$

Taking logs on both sides and linearizing, one gets (specializing to the CARA utility case):[33]

$$\hat{Y}_t = -\frac{1}{\gamma y_h'(Y)}r_t + \frac{p_{l|h}e^{\gamma(y_h-y_l)}\left(\frac{y_l'(Y)}{y_h'(Y)}\right) + p_{h|h}}{p_{l|h}e^{\gamma(y_h-y_l)} + p_{h|h}}\hat{Y}_{t+1}$$

A natural way to generate countercyclical income risk in this setting would be to posit $y_l'(Y) > 1 > y_h'(Y)$.

[33]More generally this equation takes the form:

$$\frac{u''(y_h)}{u'(y_h)}y_h'(Y)\hat{Y}_t = r_t + \frac{p_{l|h}u''(y_l)y_l'(Y) + p_{h|h}u''(y_h)y_h'(Y)}{p_{l|h}u'\left(y_l\right) + p_{h|h}u'\left(y_h\right)}\hat{Y}_{t+1}$$

Since low income agents are constrained, this also has the effect of increasing the cyclical sensitivity of HTM income. A lower value of $y_h'(Y)$ increases the sensitivity of GDP to real interest rates. But this sensitivity increases not because income risk is countercyclical, but because the income of HTMs is more cyclically sensitive (thus, the income of unconstrained agents is less sensitive). To see this, consider the limit where $p_{l|h} \to 0$ and the each agent's type is fixed (as in our model with $\eta > 0$ and $\sigma_y \to 0$). In the limit, idiosyncratic income risk is zero, and thus $y_h'(Y) < 1$ clearly does not generate countercyclical idiosyncratic income risk. However, it still generates a higher interest rate elasticity of GDP as can be seen from the equation above. This example (which closely mimics the environment in Bilbiie (2017a) and Werning (2015)) suggests that the cyclicality of income risk is not the primary force affecting the interest rate elasticity of aggregate spending and instead it is the differential sensitivity of constrained and unconstrained agents' income which is key. However, in their framework it is difficult to distinguish these two features since they are governed by the same parameters. In contrast, our framework allows us to independently vary the cyclicality of risk and HTM income. Doing so reveals that cyclicality of HTM income, not cyclicality of risk, affects the interest rate sensitivity of aggregate spending.

6.2 "Discounted/explosive" Euler equations

To proceed further it is instructive to linearize (30) around steady state to yield:

$$\hat{Y}_t = -\widetilde{\gamma}^{-1}\left(i_t - \pi_{t+1}\right) + \widetilde{\Theta}\hat{Y}_{t+1} - \widetilde{\Lambda}\hat{\mu}_{t+1} \tag{31}$$

where $\widetilde{\Theta} = \Xi\Theta + 1 - \Xi$ and $\widetilde{\Lambda} = \Xi\Lambda$. As discussed above, when the incomes of constrained and unconstrained households are equally sensitive to aggregate income ($\chi = 1$), GDP moves one for one with unconstrained households' consumption ($\Xi = 1$), and the aggregate Euler equation is unaffected by the introduction of some HTM agents. In particular, $\widetilde{\Theta} = \Theta$ and whether the Euler equation is "discounted" or "explosive" depends solely on whether income risk is procyclical, countercyclical or acyclical and is unaffected by MPC heterogeneity.

Suppose instead that HTM incomes are less cyclically sensitive ($\chi < 1$), so that GDP moves less than one for one with c_t^u ($\Xi < 1$). In this case $\widetilde{\Theta}$ is just the weighted average of Θ and 1. Thus, (31) is similar to the Euler equation without MPC heterogeneity (20), except that the coefficient on future GDP is now closer to 1. Thus, in this case, MPC heterogeneity reduces the effect of idiosyncratic risk on aggregate outcomes, pulling the HANK economy back towards the RANK benchmark. If risk is procyclical, MPC heterogeneity makes the Euler equation "less discounted", increasing the sensitivity of current spending to far future changes in interest rates, making determinacy under a peg less likely and so forth. Similarly, if risk is countercyclical, MPC heterogeneity makes the Euler equation less "explosive". While MPC heterogeneity makes a discounted Euler equation "less discounted", and a explosive Euler equation "less explosive", it cannot turn a discounted Euler equation into an explosive one, or vice versa.

However, the effect of MPC heterogeneity is reversed if HTM incomes are more cyclically sensitive $\chi > 1$, so that GDP moves more than one for one with c_t^u, ($\Xi > 1$). In this case, $\Xi > 1$ drives $\widetilde{\Theta}$ further away from 1, pushing the HANK economy further away from the RANK benchmark. If risk is procyclical, MPC heterogeneity now makes the Euler equation even more discounted; if risk is countercyclical, it makes the Euler equation even more explosive. Again, in both cases, MPC heterogeneity amplifies the deviation from the RANK benchmark but does not change the direction of this deviation.

MPC heterogeneity and determinacy of equilibrium Recall that 3 showed that determinacy in our HANK economy without MPC heterogeneity depended on whether monetary policy satisfied the risk augmented Taylor principle (25). Since the aggregate Euler equation with MPC heterogeneity (31) has the same form as (20), it is immediate that the augmented Taylor principle in this case takes the form:

$$\Phi_\pi > 1 + \frac{\widetilde{\gamma}}{\kappa}\left[\frac{\left(1-\tilde{\beta}\right)^2}{\left(1-\tilde{\beta}\right)+\widetilde{\gamma}\tilde{\beta}\widetilde{\Lambda}}\right]\left(\widetilde{\Theta}-1\right) = 1 + \frac{\gamma}{\kappa}\left[\frac{\left(1-\tilde{\beta}\right)^2}{\left(1-\tilde{\beta}\right)+\gamma\tilde{\beta}\Lambda}\right](\Theta-1)$$

where the second equality holds because $\tilde{\gamma} = \gamma/\Xi$, $(\widetilde{\Theta}-1) = \Xi(\Theta-1)$ and $\widetilde{\Lambda} = \Xi\Lambda$. That is, the introduction of MPC heterogeneity does not change the determinacy properties of equilibrium, even if $\chi \neq 1$. The cyclicality of income risk remains the most important feature affecting determinacy. Intuitively, MPC heterogeneity alters the static relation between the spending of unconstrained households and GDP, but does not affect the dynamic relationship between the spending of unconstrained households today and in the future. Thus, while χ might be very important for other questions, it **does not** affect determinacy given the cyclicality of income risk.

MPC heterogeneity and forward guidance puzzle We have seen in 5.1 that a substantial degree of procyclical risk is necessary to resolve the FGP - i.e., to make the sensitivity of current spending to future changes in interest rates decreasing, rather than increasing, in the horizon of the interest rate change. With MPC heterogeneity, it remains true that procyclical risk, $\Theta < 1$ is necessary to solve the puzzle. Indeed if $\chi < 1$, even more procyclical risk is required to solve the puzzle. While $\chi < 1$ reduces the sensitivity of GDP to interest rates $\widetilde{\gamma}^{-1}$, it does so at all horizons. Thus, a lower $\widetilde{\gamma}^{-1}$ does not necessarily mean that interest rates far in the future are less effective *relative* to interest rates today. Indeed, suppose that risk was sufficiently procyclical (Θ sufficiently below 1) that absent HTM agents, the FGP was resolved. Adding HTM agents with less cyclically sensitive income, $\chi < 1$ would actually re-awaken the FGP by bringing the effective $\widetilde{\Theta}$ closer to 1. Conversely, if risk was slightly procyclical but not enough to solve the FGP (Θ slightly below 1), introducing hand-to-mouth agents with very cyclical incomes ($\chi > 1$) moves the economy further away from the RANK benchmark, potentially solving the FGP.

These results might seem to contradict the findings in Bilbiie (2017a) who argues that $\chi < 1$ is the key to solve the FGP. More precisely, Bilbiie (2017a) argues that a necessary condition for his stylized TANK model to solve the FGP is that income of HTMs is less cyclically sensitive than that of unconstrained agents. In contrast, we have just shown that in an economy with procyclical risk, $\chi < 1$ actually works against solving the FGP while $\chi > 1$ helps. The reason for this apparent discrepancy is that in Bilbiie (2017a) $\chi < 1$ is a necessary condition for idiosyncratic income risk to be procyclical. In his model, the only risk faced by unconstrained households is the risk of becoming HTM. Thus, risk is lower in recessions iff HTM income falls less than aggregate income in recessions: the same parameter χ determines whether risk is pro- or countercyclical, and whether HTM income is more or less cyclicaly sensitive than the income of the unconstrained. Our model, which does not impose any necessary relation between these two cyclicalities, makes it clear that it is the cyclicality of risk, *not* the cyclical sensitivity of HTM income, which primarily determines whether the model features the FGP or not.

More generally, our point is that MPC heterogeneity (and the cyclicality of HTM income) and idiosyncratic income risk (and its cyclicality) are two conceptually distinct features of an incomplete markets economy, which change the behavior of the NK model in different ways. To be clear, we do not wish the reader to interpret this as meaning that these features are unrelated, and that risk cyclicality and HTM income cyclicality are free parameters which can be chosen independently. Any structural model will make a prediction about the relation between these two forces. Even in our model, different specification of fiscal policies will imply different relationships between Θ and χ. For example, the transfer policy described in the beginning of section 6 implied no relation between Θ and χ. Alternatively, consider the case in which the fiscal authority only taxed unconstrained households using the proportional tax $\boldsymbol{\tau}(Y)$ to raise resources for the transfer to constrained agents. In this case, the fiscal authority is lowering the effective risk faced by unconstrained agents (lowering Θ relative to the earlier case) and at the same time making $\chi > 1$.

But although the cyclicality of risk and the cyclical sensitivity of HTM income are undoubtedly related, there is no consensus on what is the relevant combination of the two even within the recent literature on analytically tractable HANK models. For example, Bilbiie (2017a) and section 3.3 of Werning (2015) feature models in which countercyclical income risk implies more cyclically sensitive HTM income, and procyclical risk implies less cyclically sensitive income. McKay et al. (2017) is a special case of the latter possibility in which constrained agents have constant income. On the other hand, Ravn and Sterk (2017b) and section 3.4 of Werning (2015) study an economy in which unemployment makes idiosyncratic risk countercyclical even though constrained agents do not have more cyclical income (in fact Ravn and Sterk (2017b) features acyclical HTM income).[34] More generally, in quantitative models which do not permit analytical solutions, both the cyclicality of income risk and cyclical sensitivity of the income of high MPC depends on a host of features: the distribution of profits across agents, fiscal policy, the types of assets agents can trade, and so forth. Since plausible models can generate different predictions about the combination of Θ and χ, it would be useful to study the distinct effect of each one in isolation. The virtue of our framework is that by dirempting the cyclicality of risk from the cyclicality of HTM income, it allows us to do precisely that.

Our framework also sheds light on what features are required in order for a HANK model to address RANK puzzles and at the same time generate amplification. For example, Bilbiie (2017a) argues that the same features that resolve the FGP also reduce the impact of a monetary policy shock. To understand his point, notice that the contemporaneous effect of a change in interest rates on GDP is given by $\widetilde{\gamma}^{-1} = \Xi/\gamma$, while the forward guidance puzzle is ameliorated when $\widetilde{\Theta} = \Xi\Theta + 1 - \Xi$ is sufficiently far below 1. As explained above, in Bilbiie (2017a), the only way to generate procyclical income risk which makes $\widetilde{\Theta} < 1$ is to assume that HTM income moves less than proportionally with GDP. This resolves the FGP but reduces the impact of monetary policy shocks, since less cyclical HTM income $\chi < 1$ reduces Ξ. In our more general environment, it is perfectly plausible to have both very cyclically sensitive HTM income ($\chi > 1$) and procyclical income risk ($\Theta < 1$). This combination resolves the FGP without reducing the contemporaneous effectiveness of monetary policy shocks.

MPC heterogeneity and fiscal multipliers In section 5.2, we showed that our HANK economy features smaller government spending multipliers in a liquidity trap, relative to the RANK benchmark, if risk is procyclical ($\Theta < 1$); countercyclical risk instead generates larger multipliers than RANK. This

[34] Ravn and Sterk (2017b) and section 3.4 of Werning (2015) do not map directly into our economy with $\Theta > 1$ and $\chi < 1$ since they feature a time varying fraction of HTM agents.

intuition carries over to an economy with MPC heterogeneity, provided that Θ is replaced with $\widetilde{\Theta}$. More precisely, in the fixed price limit, the response of output to a balanced budget increase in government purchases in a liquidity trap lasting T periods is :

$$\frac{\partial \hat{Y}_t}{\partial g} = \begin{cases} \widetilde{\Theta}^{T-t-1} & \text{for } 0 \leq t < T \\ 0 & \text{else} \end{cases}$$

Similar to our characterization of the FGP, varying the cyclical sensitivity of HTM income can change the *magnitude* of the difference between HANK and RANK multipliers, but not the *sign* of the difference. A HANK economy with procyclical income risk $\Theta < 1$ always has a smaller balanced budget multiplier than RANK; larger values of χ reduce the multiplier further, while lower χ brings the multiplier closer towards the RANK benchmark.[35] These results broadly carry over to the case where prices are sticky rather than rigid. Figure 4 plots fiscal multipliers in an economy hit by a 5 period liquidity trap for various values of χ and Θ.[36] The left panel features procyclical risk ($\Theta = 0.8$). Absent MPC heterogeneity our HANK economy features a lower balanced budget multiplier than the RANK economy (the blue curve); this HANK multiplier is identical to the middle red curve which depicts the multiplier with MPC heterogeneity when $\chi = 1$.[37] As in the fixed price case, a lower χ moves the multipliers closer to the RANK values (upper red curve) while higher χ further reduces the multiplier (lower red curve). The rightmost panel depicts the case with countercyclical risk ($\Theta = 1.1$). In this case, our HANK economy absent MPC heterogeneity (middle red curve) features a higher multiplier than RANK (blue curve). Again, lower χ dampens the difference relative to RANK (lower red curve) but the multiplier still lies above RANK in this case.

The middle panel depicts the case of acylical risk $\Theta = 1$. With fixed prices, the HANK and RANK multipliers both equalled 1 in this case. With sticky prices, the RANK multiplier (blue curve) is greater than 1 because of the expected inflation channel: higher inflation lowers real rates boosting spending. Since lower χ reduces the sensitivity of aggregate output to real interest rates, it dampens the expected inflation channel resulting in a lower multiplier than RANK (lower red curve). A higher value of χ instead increases the sensitivity of output to real interest rates, strengthening the expected inflation channel and the multiplier (upper red curve).

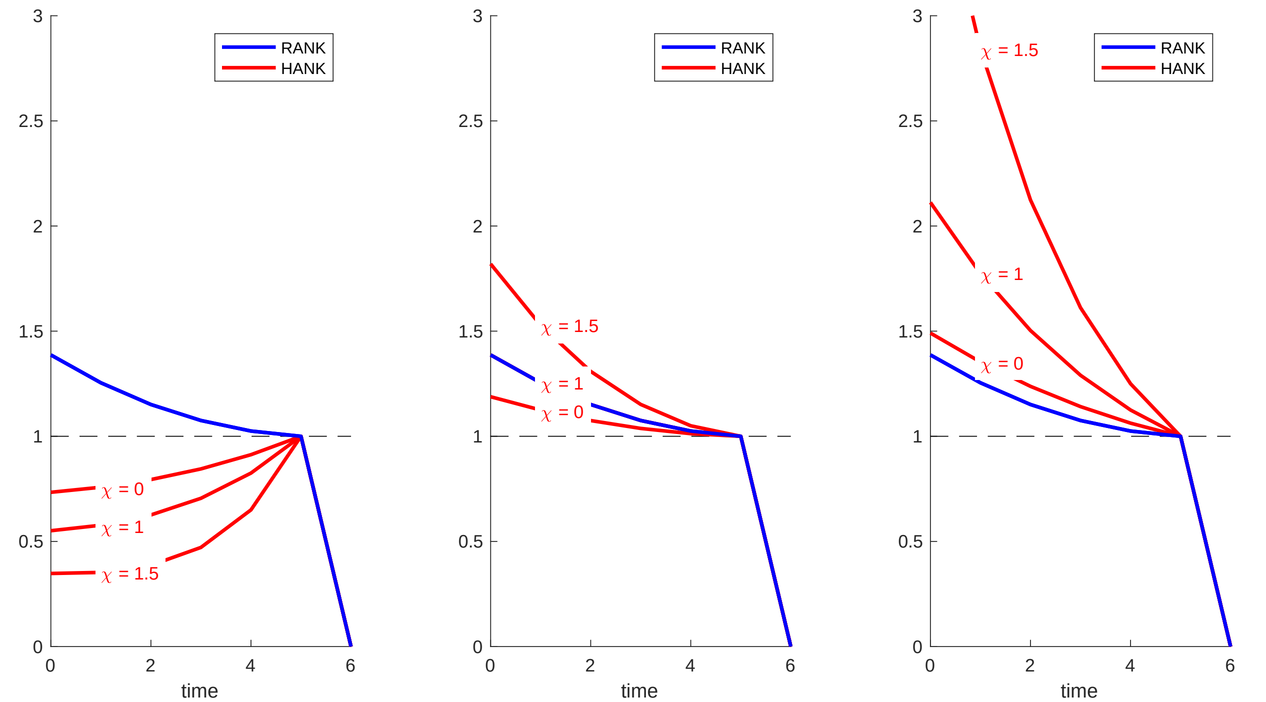
6.3 Precautionary savings channel

The only thing left to explain in (31) is $\widetilde{\Lambda} = \Xi\Lambda$. Recall that Λ denotes the precautionary savings channel of monetary policy described in Section 4. When monetary policy is tight, the MPC of unconstrained agents is high $\mu_{t+1} > 0$, and a given amount of income risk translates into higher consumption risk, a stronger precautionary saving motive, and lower current spending. With MPC heterogeneity, how this fall in c_t^u translates into a fall in GDP depends on Ξ. With $\Xi > 1$, current GDP is more sensitive to c_t^u, and thus more sensitive to future μ_t's. Again, the introduction of MPC heterogeneity tends to amplify the

[35] As is well known, in an economy with HTM agents, the deficit financed multiplier can be higher than the balanced budget multiplier (Gali et al., 2005). Introducing HTM agents also generates a transfer multiplier which was absent in our benchmark without HTM agents. See Mehrotra (2017) for details. While we abstract from these features since they have already been studied extensively, they confirm our general point that MPC heterogeneity has very different effects than the cyclicality of income risk.

[36] Following Campbell and Mankiw (1989), we set $\eta = 0.5$.

[37] Recall that when $\chi = 1$, aggregate outcomes are unaffected by the introduction of HTM households.



(a) Procyclical Risk ($\Theta = 0.8$) **(b)** Acyclical Risk ($\Theta = 1$) **(c)** Countercyclical Risk ($\Theta = 1.1$)

Figure 4. Fiscal Multipliers $\frac{d\hat{Y}_t}{dg}$ given a 5 period liquidity trap.

deviation from RANK in this aspect, but only when $\chi > 1$. When $\chi < 1$, the introduction of HTM agents actually reduces the extent to which uninsurable risk changes outcomes relative to RANK - and when $\chi = 1$, the introduction of HTM agents has no effect on the precautionary savings channel of monetary policy.

7 The pervasive importance of fiscal policy in HANK models

In our analysis, fiscal policy is a key force mediating the effect of monetary policy on equilibrium outcomes. Procyclical income risk makes indeterminacy less likely, weakens the forward guidance puzzle and reduces the government spending multiplier in a liquidity trap. Countercyclical risk makes indeterminacy more likely, worsens the forward guidance puzzle and increases the fiscal multiplier in a liquidity trap. Holding fiscal policy fixed, the cyclicality of income risk depends on structural characteristics of the economy: the cyclicality of real wages, how unemployment risk behaves in a recession, etc.[38] But holding these structural features fixed, as discussed in section 3.4, fiscal policy can change the cyclicality of income risk or even wholly flip the sign, drastically changing the transmission mechanism of monetary policy.

Just as fiscal policy can determine the cyclicality of income risk, it also determines the cyclical sensitivity of income of HTM agents. As the previous section showed, this cyclical sensitivity affects the contemporaneous response of GDP to interest rate changes, as well as the strength of the FGP and so on.

[38]Indeed,as we have shown above, whereas McKay et al. (2015), McKay et al. (2017) present a HANK economy with procyclical risk, Ravn and Sterk (2017b), Challe (2017) present one with countercyclical risk resulting in very different predictions.

Again, holding fiscal policy fixed, the cyclical sensitivity of HTM income depends on structural features of the economy such as the distributions of profits, the cyclicality of earnings and so on. However, whatever the structural features of the economy, an appropriately designed tax-transfer program can make the income of HTM agents more or less cyclically sensitive. Thus, by affecting both the cyclicality of income risk and cyclical sensitivity of the income of constrained agents, fiscal policy determines the way in which a HANK economy deviates from the corresponding RANK benchmark.

In order to focus on these particular interactions between fiscal and monetary policy, we purposely abstracted from two other important interactions. The first concerns whether fiscal policy adjusts surpluses to remain solvent along any price path (Leeper, 1991). Whether fiscal policy is active or passive in this sense crucially determines the effects of monetary policy (Sims, 2011; Cochrane, 2017a). We abstract from this issue by assuming that the government always adjusts lump sum taxes/transfers to ensure solvency. Even in this case, fiscal policy, by determining Θ, can affect everything. More generally these issues associated with the FTPL, and the "$\Theta\chi$" monetary-fiscal interactions which have been our focus, while logically distinct, may interact. For example, if the government had limited ability to vary lump-sum taxes and instead adjusted labor income taxes τ_t to ensure solvency, these movements in τ_t could change the cyclicality of income risk affecting the properties of equilibrium.

The other type of monetary-fiscal interaction arises in models where incomplete markets break Ricardian Equivalence, e.g. Kaplan et al. (2016). Changes in real interest rates may necessitate changes in fiscal surpluses in order to satisfy the government budget constraint. For example, higher real interest rates would tend to worsen the government's fiscal position which might be met by a cut in public transfers T (an increase in fiscal surpluses). If all agents were unconstrained, such a cut would have no effect on aggregate demand, but in the presence of hand-to-mouth agents, the cut in transfers would further reduce spending and output, amplifying the effect of a change in monetary policy. This is the main channel emphasized in Kaplan et al. (2016). In addition to these two types of fiscal-monetary interactions, we identify an additional dimension through which fiscal policy affects monetary policy – by determining Θ and χ.

8 Conclusion

The fast growing literature on HANK economies suggests that monetary policy works differently in such environments, relative to the standard RANK economy. Much of this literature has relied on computational methods which makes it hard (though not impossible) to understand precisely which features of incomplete markets drive the differences between particular HANK and RANK economies. Our goal has been to shed light on this question. To this end we presented a general HANK economy which can be solved in closed form. We are certainly not the first to do this, but our strategy complements the approaches pursued in the theoretical literature on HANK models in two ways. First, we do not rely on the zero liquidity limit. This allowed us to uncover an additional dimension through which monetary policy affects aggregate demand: tighter monetary policy increases the sensitivity of consumption to individual income shocks, raising consumption risk for a given level of income risk and reducing aggregate demand via the precautionary savings channel. This channel would be absent in a zero liquidity economy in which consumption risk is trivially the same as income risk. Second, our framework nests several different specifications of HANK economies in the theoretical and computational literature so far, and explains how differences in their conclusions arise from the differences in the cyclicality of income risk and the cyclical sensitivity of the income of constrained agents in these economies. In particular, we can distinguish between the cyclicality of risk and heterogeneous cyclical sensitivities of income (who loses the most in a recession). Other recent work on analytically tractable HANK models either abstracts from one of these forces altogether or imposes a very tight connection between them. By clearly delineating these two forces, our framework allows us to trace out their distinct effects on macroeconomic outcomes.

Our distinction between precautionary savings (and the cyclicality of risk) on the one hand, and MPC heterogeneity (and the cyclicality of HTM income) on the other, has revealed that it is the former which matters more for the topics we have focused on - determinacy, the forward guidance puzzle, and so forth. This by no means implies that MPC heterogeneity is irrelevant. Indeed, we have shown that it can change the contemporaneous sensitivity of aggregate output to interest rates, which is unaffected by the cyclicality of income risk. More generally, MPC heterogeneity would also affect the aggregate demand effects of redistributive policies, the effects of the supply of government debt, and so forth.

While we have relied on a analytically tractable model to derive closed form results, we believe that our qualitative results would hold more generally. Indeed Auclert et al. (2017) numerically confirm our finding that procyclical risk makes determinacy more likely, in their CRRA economy with borrowing constraints. Since the properties of equilibrium depend so closely on two low dimensional statistics - the cyclicality of income risk and the cyclicality of HTM income - measuring these objects empirically might help us better understand the monetary transmission mechanism in reality.[39]

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[39] A caveat is that, as mentioned above, the relevant measure of cyclicality is not the *correlation* between aggregate output and idiosyncratic risk - as measured by Storesletten et al. (2004), Guvenen et al. (2014), among others - but the *effect* of an increase in aggregate output on idiosyncratic risk, holding other variables fixed.

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Appendix

A Deriving the consumption decision rule

Given a path of real interest rates $\{r_t\}$ and aggregate output $\{y_t\}$, household i's problem is:

$$\begin{aligned} \max_{\{c_t^i\}} \quad & -\gamma^{-1} E_0 \sum_{t=0}^{\infty} \beta^t e^{-\gamma c_t^i} \\ \text{s.t.} \quad & c_t^i + \frac{1}{1+r_t} a_{t+1}^i = a_t^i + y_{i,t} & (32) \\ & y_t^i \sim N\Big(y_t, \sigma_t^2(y_t)\Big) & (33) \end{aligned}$$

where $a_t^i = A_t^i / P_t$ denotes the real value of household i's wealth at the beginning of date t. The optimal choices of the household can be summarized by the standard euler equation:

$$e^{-\gamma c_t^i} = \beta\left(1+r_t\right) E_t e^{-\gamma c_{t+1}^i} \tag{34}$$

Taking logs on both sides, the equation above can be written as:

$$-\gamma c_t^i = \ln \beta\left(1+r_t\right) + \ln \mathbb{E}_t e^{-\gamma c_{t+1}^i} \tag{35}$$

Next, we guess that the consumption decision rule of household i takes the form:

$$c_t^i = \mathcal{C}_t + \mu_t\left(a_t^i + y_t^i\right) \tag{36}$$

where $\mathcal{C}_t$ and μ_t are deterministic processes that are common across all households. Given this guess, we can use the budget constraint (32) to write:

$$a_{t+1}^i = \left(1+r_t\right)\left(1-\mu_t\right)\left(a_t^i + y_t^i\right) - \left(1+r_t\right)\mathcal{C}_t \tag{37}$$

Using equation (37), one can express consumption at date $t+1$ as:

$$\begin{aligned} c_{t+1}^i &= \mathcal{C}_{t+1} + \mu_{t+1}\left(a_{t+1}^i + y_{t+1}^i\right) \\ &= \mathcal{C}_{t+1} + \mu_{t+1}\left[\left(1+r_t\right)\left(1-\mu_t\right)\left(a_t^i + y_t^i\right) - \left(1+r_t\right)\mathcal{C}_t + y_{t+1}^i\right] \end{aligned} \tag{38}$$

Then it is straightforward to see that:

$$E_t\left[-\gamma c_{t+1}^i\right] = -\gamma \mathcal{C}_{t+1} - \gamma \mu_{t+1}\left[\left(1+r_t\right)\left(1-\mu_t\right)\left(a_{i,t} + y_{i,t}\right) - \left(1+r_t\right)\mathcal{C}_t + y_{t+1}\right] \tag{39}$$

$$E_t\left(-\gamma c_{t+1}^i - E_t\left[-\gamma c_{t+1}^i\right]\right)^2 = \gamma^2 \mu_{t+1}^2 \sigma_{y,t+1}^2 \tag{40}$$

and using the property of log-normals:

$$\ln \mathbb{E}_t e^{-\gamma c_{t+1}^i} = -\gamma \mathcal{C}_{t+1} - \gamma \mu_{t+1}\left[\left(1+r_t\right)\left(1-\mu_t\right)\left(a_{i,t} + y_{i,t}\right) - \left(1+r_t\right)\mathcal{C}_t + y_{t+1}\right] + \frac{\gamma^2 \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \tag{41}$$

Using this in the Euler equation (35) and matching coefficients:

$$\begin{aligned}\mu_t &= \frac{\mu_{t+1}(1+r_t)}{1+\mu_{t+1}(1+r_t)} && (42)\\ \mathcal{C}_t\left[1+\mu_{t+1}(1+r_t)\right] &= -\frac{1}{\gamma}\ln\beta(1+r_t)+\mathcal{C}_{t+1}+\mu_{t+1}y_{t+1}-\frac{\gamma\mu_{t+1}^2\sigma_{y,t+1}^2}{2} && (43)\end{aligned}$$

which verifies the guess. Next, solving (43) forwards and using (42) yields equation (12) in the main text:

$$\mathcal{C}_t=\sum_{s=1}^{\infty}Q_{t+s|t}\frac{\mu_t}{\gamma\mu_{t+s}}\ln\left[\frac{1}{\beta(1+r_{t+s-1})}\right]+\mu_t\sum_{s=1}^{\infty}Q_{t+s|t}y_{t+s}-\frac{\gamma\mu_t}{2}\sum_{s=1}^{\infty}Q_{t+s|t}\mu_{t+s}\sigma_{y,t+s}^2 \tag{44}$$

where $Q_{t+s|t}=\prod_{k=0}^{s-1}\left(\frac{1}{1+r_{t+k}}\right)$. If real interest rates are constant at a level $r>0$, then (42) implies that

$$\mu_t=\mu=\frac{r}{1+r}>0\ ,\ \forall t \tag{45}$$

which confirms the claim in Corollary **??**.

B Deriving the Aggregate Euler Equation

In order to derive the aggregate Euler equation, we start with the individual consumption decision rules. Since μ_t and $\mathcal{C}_t$ do not have i superscripts, i.e. they are the same across all households, independent of wealth of income. Thus, we can linearly aggregate this economy to get an *aggregate consumption function*:

$$\begin{aligned}c_t &= \int c_t^i di=\mathcal{C}_t+\mu_t\int\left(a_t^i+y_t^i\right)di\\ &= \mathcal{C}_t+\mu_t\left(\frac{B_t}{P_t}+y_t\right) && (46)\end{aligned}$$

where we have used asset market clearing $\left(\frac{B_t}{P_t}=\int a_t^i di\right)$ and the fact that $y_t=\int y_t^i di$ in the second line. Then, using (46) in (43):

$$\begin{aligned}\left[c_t-\mu_t\left(\frac{B_t}{P_t}+y_t\right)\right]\left[1+\mu_{t+1}(1+r_t)\right] &= -\frac{1}{\gamma}\ln\beta(1+r_t)+c_{t+1}-\mu_{t+1}\left(\frac{B_{t+1}}{P_{t+1}}+y_{t+1}\right)+\mu_{t+1}y_{t+1}\\ &\quad-\frac{\gamma\mu_{t+1}^2\sigma_{y,t+1}^2}{2}\end{aligned}$$

Next, using (8) and (42), we can rewrite the equation above as:

$$\begin{aligned}\left[c_t-\mu_t\left(\mathcal{S}_t+\frac{B_{t+1}}{P_{t+1}}\frac{1}{1+r_t}+y_t\right)\right]\left[\frac{1}{1-\mu_t}\right] &= -\frac{1}{\gamma}\ln\beta(1+r_t)+c_{t+1}-\mu_{t+1}\left(\frac{B_{t+1}}{P_{t+1}}+y_{t+1}\right)\\ &\quad+\mu_{t+1}y_{t+1}-\frac{\gamma\mu_{t+1}^2\sigma_{y,t+1}^2}{2}\end{aligned}$$

Recall that $Y_t = \mathcal{S}_t + G_t + y_t$ and in general equilibrium, $c_t + G_t = Y_t$. Combining this information with the information above:

$$Y_t = -\frac{1}{\gamma} \ln \beta \left(1 + r_t\right) + Y_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} + G_t - G_{t+1} \tag{47}$$

This is the same as equation (14) in the main text and (28) is the linearized version of this equation.

C The 4 Equation HANK Model

In this section we present the linearized model. Recall that the aggregate dynamics of our HANK economy can be fully described by:

$$\begin{aligned}
Y_t &= -\frac{1}{\gamma} \ln \beta \left(1 + r_t\right) + Y_{t+1} - \frac{\gamma \mu_{t+1}^2 \boldsymbol{\sigma}^2(Y_t)}{2} + G_t - G_{t+1} && (48) \\
\mu_t &= \frac{\mu_{t+1}\left(1 + r_t\right)}{1 + \mu_{t+1}\left(1 + r_t\right)} && (49) \\
\Xi \Pi_t \left(\Pi_t - 1\right) &= 1 - \theta \left(1 - x_t^{\frac{1-\alpha}{\alpha}}\right) + \Xi \left(\Pi_{t+1} - 1\right) \Pi_{t+1} \left[\frac{1}{1 + r_t} \frac{x_{t+1}}{x_t}\right] && (50) \\
1 + i_t &= (1 + r) \Pi_t^{\Phi_\pi} && (51)
\end{aligned}$$

where $1 + r_t = \frac{1+i_t}{\Pi_{t+1}}$ and $Y_t = x_t - x_t^{\frac{1}{\alpha}}$. Next, we linearize the model. In what follows, i_t, π_t and $\hat{\mu}_t$ denote the log-deviations of $1 + i_t, \Pi_t$ and μ_t from their steady state values $1 + i = 1 + r, \Pi = 1$ and $\mu = \frac{r}{1+r}$ respectively. $\hat{Y}_t$ and $\hat{G}_t$ denote the deviations in levels of Y_t and G_t from the steady state levels $Y = \left(\frac{1}{\theta - 1}\right)\left(1 - \frac{1}{\theta}\right)^{\frac{1}{1-\alpha}}$ and G. The linearized model is given by:

$$\begin{aligned}
\hat{Y}_t &= \Theta \hat{Y}_{t+1} - \gamma^{-1}\left(i_t - \pi_{t+1} - \rho_t\right) - \Lambda \hat{\mu}_{t+1} + \hat{G}_t - \hat{G}_{t+1} && (52) \\
\hat{\mu}_t &= \tilde{\beta}\left(\hat{\mu}_{t+1} + i_t - \pi_{t+1}\right) && (53) \\
\pi_t &= \tilde{\beta} \pi_{t+1} + \kappa \hat{Y}_t && (54) \\
i_t &= \Phi_\pi \pi_t && (55)
\end{aligned}$$

where $-\rho_t$ is the log-deviation of β from steady state, $\Lambda = \gamma \mu \boldsymbol{\sigma}^2(Y^*)$, $\Theta = 1 - \frac{\gamma \mu^2}{2} \frac{d\boldsymbol{\sigma}^2(Y^*)}{dY_t}$, $\tilde{\beta} = \frac{1}{1+r}$ is the inverse of the steady state real interest rate, and $\kappa = \frac{\theta - 1}{\Xi}\left[\frac{\theta(1-\alpha)}{1-\theta(1-\alpha)}\right]\left(\frac{\theta}{\theta - 1}\right)^{\frac{\alpha}{1-\alpha}}$. Notice that as $\Xi \to \infty$ (prices become perfectly rigid) $\kappa \to 0$.

C.1 The 3 equation RANK model

The standard 3 equation RANK model is a special case of our 4 equation HANK model. In the case of a representative agent model, $\boldsymbol{\sigma}^2(Y) = \frac{d\boldsymbol{\sigma}^2(Y)}{dY} = 0$ and $\tilde{\beta} = \beta$. Thus, in the RANK model, $\Lambda = 0$ and $\Theta = 1$.

Thus, we can write the system as:

$$\hat{Y}_t = \hat{Y}_{t+1} - \gamma^{-1}\left(i_t - \pi_{t+1} - \rho_t\right) + \hat{G}_t - \hat{G}_{t+1} \quad (56)$$

$$\pi_t = \tilde{\beta}\pi_{t+1} + \kappa\hat{Y}_t \quad (57)$$

$$i_t = \Phi_\pi \pi_t \quad (58)$$

Notice that the dynamics of $\hat{\mu}_t$ given by $\hat{\mu}_t = \tilde{\beta}\left(\hat{\mu}_{t+1} + i_t - \pi_{t+1}\right)$ no longer affect the dynamics of $\hat{Y}_t$ and π_t. Thus, we can ignore that equation in the RANK model.

C.2 Determinacy properties of the RANK model under a peg

It is commonly known that if the monetary authority follows a nominal interest rate peg, $\Phi_\pi = 0$ then the standard RANK model features local indeterminacy. In other words, with $\Phi_\pi = 0$ there are multiple bounded path of $\hat{Y}_t$ and π_t which satify equations (56)-(58). More generally, as long as $|\Phi_\pi| < 1$, the standard RANK model features local indeterminacy. See Sargent and Wallace (1975), Bullard and Mitra (2002) and Galí (2015) for a detailed exposition. This indeterminacy is generally associated with unanchored inflation. If prices are sticky $\kappa > 0$, the indeterminacy in prices also manifests itself in output. However, if prices are perfectly rigid, indeterminacy under a nominal peg manifests only in output since prices cannot move. To see this, notice that with perfectly rigid prices, the RANK model can be written as (with $\hat{G}_t = 0$ wlog):

$$\hat{Y}_t = \hat{Y}_{t+1} \quad (59)$$

$$\pi_t = 0 \quad (60)$$

As can be clearly seen, any constant level of output is consistent with a bounded equilibrium in this case. With fixed prices, a fixed nominal interest rate translates into a fixed real interest rate. In this extreme case, output is demand determined expectations of higher income in the future are self-fulfilling and raise the level of income today by the same amount.

D Determinacy in HANK

Setting $\hat{G}_t = \rho_t = 0$, equations (52)-(55) can be written in matrix form as

$$\underbrace{\begin{bmatrix} \Theta & \frac{1}{\gamma} & -\Lambda \\ 0 & \tilde{\beta} & 0 \\ 0 & -\tilde{\beta} & \tilde{\beta} \end{bmatrix}}_{A} \begin{bmatrix} \hat{Y}_{t+1} \\ \hat{\pi}_{t+1} \\ \hat{\mu}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{\Phi_\pi}{\gamma} & 0 \\ -\kappa & 1 & 0 \\ 0 & -\tilde{\beta}\Phi_\pi & 1 \end{bmatrix}}_{B} \begin{bmatrix} \hat{Y}_t \\ \hat{\pi}_t \\ \hat{\mu}_t \end{bmatrix}$$

and so

$$A^{-1}B = \begin{bmatrix} \frac{1}{\theta} + \frac{\kappa\left(\gamma^{-1}-\Lambda\right)}{\beta\theta} & \frac{\left(\gamma^{-1}-\Lambda\right)\left(\Phi_\pi-\beta^{-1}\right)}{\theta} & -\frac{\Lambda}{\beta\theta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} - \Phi_\pi & \frac{1}{\beta} \end{bmatrix}$$

The characteristic polynomial of this system is:

$$\mathcal{P}(z) = \frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}^2\Theta} - z^3 + \frac{\tilde{\beta}^2 + 2\tilde{\beta}\Theta + \tilde{\beta}\kappa(\gamma^{-1}-\Lambda)}{\tilde{\beta}^2\Theta}z^2 - \frac{2\tilde{\beta} + \tilde{\beta}\kappa\Phi_\pi(\gamma^{-1}-\Lambda) + \Theta + \kappa\gamma^{-1}}{\tilde{\beta}^2\Theta}z \tag{61}$$

Determinacy requires that all the roots of $\mathcal{P}$ lie within the unit circle. Clearly we have $\lim_{z\to-\infty}\mathcal{P}(z) = +\infty$, $\lim_{z\to+\infty}\mathcal{P}(z) = -\infty$. Under the assumption that $\gamma^{-1} > \Lambda$, $\mathcal{P}(z) > 0\ \forall z \leq 0$. Thus, determinacy requires that $\mathcal{P}(1) > 0$. Otherwise, the polynomial would have at least one real root within the unit circle. Thus we need

$$\mathcal{P}(1) = \frac{(1-\Theta)\left(1-\tilde{\beta}\right)^2}{\tilde{\beta}^2\Theta} + \frac{\kappa\gamma^{-1}\left(1-\tilde{\beta}\right) + \tilde{\beta}\kappa\Lambda}{\tilde{\beta}^2\Theta}(\Phi_\pi - 1) > 0$$

which implies

$$\Phi_\pi > 1 + \frac{1}{\kappa}\left[\frac{\gamma\left(1-\tilde{\beta}\right)^2}{\left(1-\tilde{\beta}\right)+\gamma\tilde{\beta}\Lambda}\right](\Theta - 1) \tag{62}$$

Under our maintained assumptions, this condition is always necessary for determinacy. Next, we show that it is sufficient for determinacy under the following assumptions.

Assumption 2. $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$, $\Lambda < \gamma^{-1}$, *and income risk is not too countercyclical, i.e.*

$$\Theta - 1 < \min\left\{\frac{(1-\tilde{\beta})(1-\tilde{\beta}^2+\gamma^{-1}\kappa) + \tilde{\beta}(\gamma^{-1}\kappa - \tilde{\beta}^2)\gamma\Lambda}{\tilde{\beta} - \frac{(1-\tilde{\beta})^2}{(1-\tilde{\beta})+\tilde{\beta}\gamma\Lambda} - \tilde{\beta}^3(1-\gamma\Lambda)}, \frac{1}{\tilde{\beta}\Lambda} - 1 + \frac{\gamma^{-1}\kappa}{\tilde{\beta}^2}, \frac{1-\tilde{\beta}+\gamma^{-1}\kappa}{\tilde{\beta} - \frac{(1-\tilde{\beta})^2}{1-\tilde{\beta}+\tilde{\beta}\gamma\Lambda}}\right\}$$

Note that when $\Lambda = 0$, this last assumption reduces to

$$\Theta - 1 < \min\left\{\frac{(1-\tilde{\beta})(1-\tilde{\beta}^2+\gamma^{-1}\kappa)}{2\tilde{\beta}-1-\tilde{\beta}^3}, \frac{1-\tilde{\beta}+\gamma^{-1}\kappa}{2\tilde{\beta}-1}\right\}$$

which is strictly positive, under our other assumptions. Thus for Λ sufficiently close to zero, our risk-adjusted Taylor principle is sufficient for determinacy even under moderately countercyclical income risk. The other assumptions in 2 are satisfied for reasonable parameter values of the discount factor β as we now show.

Lemma 1. *If* $\beta > \frac{\sqrt{5}}{2} - \frac{1}{2}$, *then* $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$ *and* $\Lambda < \gamma^{-1}$.

Proof. Recall that we define $\tilde{\beta} = \frac{1}{1+r}$, $\Lambda = \gamma\left(\frac{r}{1+r}\right)^2\sigma_y^2$, and r solves

$$\frac{(1+r)^2\ln\left[\frac{1}{\beta(1+r)}\right]}{\gamma^2 r^2} = \frac{1}{2}\sigma_y^2$$

It is immediate that $\tilde{\beta} \in (\beta, 1)$, so $\tilde{\beta} > \beta > \frac{\sqrt{5}}{2} - \frac{1}{2}$ and $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$. Using the definition of Λ and the fact that $r > 0$, and rearranging,

$$\gamma\Lambda < 2\ln\left(\frac{1}{\beta}\right) < 2\ln\left(\frac{2}{\sqrt{5}-1}\right) < 1$$

So we are done. $\square$

With these assumptions in hand, we can provide sufficient conditions for determinacy. Lemma 2 provides sufficient conditions for a general cubic polynomial to have all its roots outside the unit circle, as required for determinacy. Lemma 3 then concludes by showing that these conditions are satisfied if Assumption 2 holds and the risk-adjusted Taylor principle (62) is satisfied.

Lemma 2. *Consider the characteristic polynomial*

$$\mathcal{P}(z) = -z^3 + A_2 z^2 + A_1 z + A_0 = (z_1 - z)(z_2 - z)(z_3 - z)$$

Suppose $A_2 > 0$,$A_1 < 0$,$A_0 > 0$. Then the following two conditions are sufficient[40] *for $\mathcal{P}$ to have three roots outside the unit circle:*

$$\begin{aligned} A_0^2 - A_0 A_2 - A_1 - 1 &> 0 \quad (63)\\ -1 + A_2 + A_1 + A_0 &> 0 \quad (64)\\ A_1 &> 1 \quad (65)\end{aligned}$$

Proof. Assume the conditions hold. Since $\mathcal{P}(1) = -1 + A_2 + A_1 + A_0 > 0$ and $\lim_{z\to+\infty} \mathcal{P}(z) = -\infty$, there is at least one real root above 1; let this be z_3. Either z_1, z_2 are complex conjugates, or they are both real. Suppose they are complex conjugates. Note that

$$(z_1 z_2 - 1)(z_2 z_3 - 1)(z_3 z_1 - 1) = A_0^2 - A_0 A_2 - A_1 - 1 > 0$$

$z_2 z_3 - 1$ and $z_3 z_1 - 1$ are complex conjugates, so their product is a positive real number. So we must have $z_1 z_2 = |z_1| = |z_2| > 1$, i.e. all eigenvalues lie outside the unit circle in this case. Suppose then that z_1, z_2 are both real. $\mathcal{P}(0) = A_0 > 0$, so $\mathcal{P}$ has either two real roots in $(0, 1)$ or none (in which case we are done, since it has no negative real roots). Suppose $z_1, z_2 \in (0, 1)$. By (64), we have

$$z_3^2 (z_1 z_2 - 1)(z_1 - z_3^{-1})(z_2 - z_3^{-1}) > 0$$

$z_3^2 (z_1 z_2 - 1) < 0$ by assumption, so we must have (letting $z_1 < z_2$ without loss of generality)

$$0 < z_1 < z_3^{-1} < z_2 < 1$$

So $z_1 z_2 z_3 < z_3^{-1} z_2 z_3 = z_2 < 1$. Since we have assumed $A_0 = z_1 z_2 z_3 < 1$, this case is ruled out. Then it must be that all eigenvalues lie outside the unit circle. $\square$

Next, we show that under assumption 2, the risk-adjusted Taylor principle (62) is sufficient to ensure that the conditions in Lemma 2 obtain.

Lemma 3. *Suppose Assumption 2 holds and Φ_π satisfies (62). Then $A^{-1}B$ has three eigenvalues outside the unit circle.*

[40] The first two conditions are also necessary.

Proof. Suppose (62) holds: then (64) holds. Given our assumptions, we have $A_0, -A_1, A_2 > 0$. It only remains to show (63) and (65). Using the definition of the characteristic polynomial in (61), some algebra yields

$$\begin{aligned} A_0^2 - A_0A_2 - A_1 - 1 \quad &= \quad \beta^{-1} \underbrace{\left(\frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}\Theta} - 1\right)}_{B_1} \underbrace{\left(\frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}^2\Theta} - \frac{\tilde{\beta}+\Theta+\kappa(\gamma^{-1}-\Lambda)}{\tilde{\beta}\Theta} + \beta\gamma(\gamma^{-1}-\Lambda)\right)}_{B_2} \\ &\quad + \underbrace{\frac{\gamma}{\beta\Theta} + \gamma\left(\frac{\gamma^{-1}\kappa}{\tilde{\beta}^2\Theta} - 1\right)\Lambda}_{B_3} > 0 \end{aligned}$$

We will show $B_1, B_2, B_3 > 0$. First take B_2. Multiplying through by the positive number $\tilde{\beta}^2\Theta$ and using the lower bound on Φ_π given by (62), we have

$$\tilde{\beta}^2\Theta B_2 = (1-\tilde{\beta})[\kappa\gamma^{-1} + 1 - \tilde{\beta}^2] + \tilde{\beta}(\gamma^{-1}\kappa - \tilde{\beta}^2)\gamma\Lambda - (\Theta - 1)\left[\tilde{\beta} - \frac{(1-\tilde{\beta})^2}{(1-\tilde{\beta}) + \tilde{\beta}\gamma\Lambda} - \tilde{\beta}^3(1-\gamma\Lambda)\right]$$

The term in square brackets is minimized when $\Lambda = 0$, in which case it equals $2\tilde{\beta} - 1 - \tilde{\beta}^3 > 0$. So it is positive, and we will have $B_2 > 0$ provided that

$$\Theta - 1 < \frac{(1-\tilde{\beta})[\kappa\gamma^{-1} + 1 - \tilde{\beta}^2] + \tilde{\beta}(\gamma^{-1}\kappa - \tilde{\beta}^2)\gamma\Lambda}{\tilde{\beta} - \frac{(1-\tilde{\beta})^2}{(1-\tilde{\beta})+\tilde{\beta}\gamma\Lambda} - \tilde{\beta}^3(1-\gamma\Lambda)}$$

which is guaranteed by Assumption 2. Next we show $B_3 > 0$. We have

$$B_3 = \frac{\gamma}{\beta} + \frac{\kappa\Lambda}{\beta^2} - \gamma\Lambda\Theta$$

which will be positive provided that

$$\Theta < \frac{1}{\tilde{\beta}\Lambda} + \frac{\gamma^{-1}\kappa}{\tilde{\beta}^2}$$

as ensured by Assumption 2. Next we show $B_1 > 0$. Given (62),

$$B_1 = \frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}\Theta} - 1 > \frac{1}{\tilde{\beta}\Theta}\left[1 + \gamma^{-1}\kappa + \left(\frac{\left(1-\tilde{\beta}\right)^2}{\left(1-\tilde{\beta}\right) + \gamma\tilde{\beta}\Lambda}\right)(\Theta - 1) - \beta\Theta\right]$$

which is positive provided that

$$\Theta - 1 < \frac{1 - \tilde{\beta} + \gamma^{-1}\kappa}{\tilde{\beta} - \frac{(1-\tilde{\beta})^2}{1-\tilde{\beta}+\tilde{\beta}\gamma\Lambda}}$$

as ensured by Assumption 2. This establishes that (63) is satisfied. To apply Lemma 2 we only need to check condition (65), i.e.

$$A_0 = \frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}^2\Theta} > 1$$

Since $B_1 = \frac{1+\gamma^{-1}\kappa\Phi_\pi}{\tilde{\beta}\Theta} - 1 > 0$ and $\tilde{\beta} \in (0,1)$, this is immediate. So we are done. □

E A Model with some hand-to-mouth agents

In this section, we augment our standard model to accommodate a mass of agents who are hand-to-mouth, i.e. for these agents $c_t^i = y_t^i$ for all t. Relative to our baseline model, we now have a fixed fraction η of agents who are hand-to-mouth while the rest are not (who we call unconstrained agents). The HTM agents have average after-tax income χy_t where y_t denotes aggregate after-tax income. Consequently, the average after-tax income of unconstrained agents is $\frac{1-\eta\chi}{1-\eta} y_t$. Proposition 1 is still valid for these unconstrained agents, except that their mean after-tax income is $\frac{1-\eta\chi}{1-\eta} y_t$ instead of just y_t as in the Proposition 1, where y_t denotes aggregate after-tax income. Hence their consumption decision rules can be described by (9):

$$c_t^i = \mathcal{C}_t + \mu_t(a_t^i + y_t^i) \text{ for } i \in (\eta, 1]$$

where μ_t is still defined as before by (42) but $\mathcal{C}_t$ is now defined as:

$$\mathcal{C}_t \left[1 + \mu_{t+1}\left(1 + r_t\right)\right] = -\frac{1}{\gamma} \ln \beta \left(1 + r_t\right) + \mathcal{C}_{t+1} + \frac{1 - \eta\chi}{1 - \eta} \mu_{t+1} y_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \tag{66}$$

Define c_t^u as the average consumption of the unconstrained agents, i.e., $c_t^u = \frac{1}{1-\eta} \int_\eta^1 c_t^i di$. As a result we can express, c_t^u as:

$$c_t^u = \mathcal{C}_t + \mu_t \left(a_t + \frac{1 - \eta\chi}{1 - \eta} y_t \right)$$

Asset market clearing is now given by $(1 - \eta)a_t = \frac{B_t}{P_t}$ since only the unconstrained agents hold assets. Imposing asset market clearing, we can write the above as:

$$c_t^u = \mathcal{C}_t + \frac{\mu_t}{1 - \eta} \left(\frac{B_t}{P_t} + (1 - \eta\chi) y_t \right)$$

Using this expression in (66)and using (8) and (42):

$$Y_t = -\frac{1 - \eta}{1 - \eta\chi} \frac{1}{\gamma} \ln \beta \left(1 + r_t\right) + Y_{t+1} - \Delta G_{t+1} + \frac{\eta\chi}{1 - \eta\chi} \Delta \mathcal{S}_{t+1} - \frac{\gamma}{2} \left(\frac{1 - \eta}{1 - \eta\chi} \right) \mu_{t+1}^2 \sigma_{y,t+1}^2$$