Global indeterminacy in HANK economies*

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Abstract

We show that in Heterogeneous-Agent New-Keynesian (HANK) economies with countercyclical risk the natural interest rate is endogenous and co-moves with output, leaving the economy susceptible to self-fulfilling fluctuations. Unlike in Representative-Agent New-Keynesian models, the Taylor principle is not sufficient to guarantee uniqueness of equilibrium in HANK if risk is even mildly countercyclical. In fact, we prove that multiple bounded-equilibria exist, no matter how strongly monetary policy responds to changes in inflation. Neither inertial rules nor rules which respond to output-gap fluctuations can resolve this indeterminacy. Instead, to implement a unique equilibrium, policy must stabilize endogenous natural rate fluctuations.

Keywords: monetary policy, incomplete markets, cyclicality of risk, local and global indeterminacy
JEL Codes: E31, E4, E5

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A central tenet in the science of monetary policy is that in response to higher inflation, the central bank should increase nominal interest rates *sufficiently* to raise the real interest rate. In the workhorse Representative-Agent New Keynesian (RANK) models that are used by central banks to inform policy, this idea is formalized as the "Taylor principle": a policy rule which raises nominal rates more than one-for-one in response to inflation ensures a unique equilibrium (provided that the effective lower bound doesn't bind). Without such an aggressive response, a self-fulfilling inflationary process may take hold: higher inflation reduces real interest rates, stimulating demand and pushing inflation even higher. Raising nominal rates more than one-for-one nips such an incipient inflation process in the bud, stabilizing the economy. Arguably, such a determination to keep inflation in check and inflation expectations anchored underlies the steep increase in policy rates around the world in response to the high inflation during the recovery from the COVID-19 recession.

The RANK models which provide the theoretical backbone for the Taylor principle abstract from inequality, market incompleteness and the distributional effects of monetary policy. In recent years, there has been growing interest among both academics and policymakers to understand how these features affect monetary transmission. The fast growing Heterogeneous Agent New-Keynesian (HANK) literature seeks to address this.[1] We contribute to this literature by asking how monetary policy should be conducted to ensure a unique equilibrium, i.e., prevent self-fulfilling fluctuations in output and inflation from taking hold. We conduct our analysis in a analytically-tractable continuous-time HANK economy and show that if income risk is even mildly countercyclical, then raising nominal rates in response to higher inflation, no matter how aggressively, does not guarantee a unique equilibrium, i.e, the Taylor principle is not sufficient for implementing a unique equilibrium. This is because, in HANK economies with countercyclical risk, the natural rate of interest is *endogenous* and co-moves with output.[2] Standard monetary policy rules, even if they satisfy the Taylor principle, allow for self-fulfilling fluctuations in the natural rate, which act as "*endogenous* demand shocks", and result in non-fundamental fluctuations in output and inflation. Thus, standard monetary policy rules cannot implement a unique equilibrium if risk is even mildly countercyclical. To ensure a unique equilibrium, monetary policy must act to prevent these endogenous fluctuations in the natural rate.

To see why a HANK economy with countercyclical risk is susceptible to "*endogenous* demand shocks", note that if income risk is countercyclical, then households face greater income risk during a downturn, relative to an expansion. Now, suppose that absent any change in fundamentals, households suddenly entertain the belief that the economy will enter a recession. As a result, they expect to face higher income risk in the future, and increase their desired level of precautionary savings to self-insure against the higher probability of future declines in consumption. This higher desire to save puts downward pressure on the *natural* interest rate. If monetary policy keeps policy rates unchanged in the face of a lower natural rate, households cut back on their current spending. The presence of nominal rigidities implies that this lower spending translates into lower output and below target in-

[1] See, for example, Kaplan et al. (2018); McKay et al. (2016); Auclert et al. (forthcoming); Acharya and Dogra (2020); Ravn and Sterk (2021); Bilbiie (forthcoming); Gornemann et al. (2016); Ahn et al. (2018) among others.

[2] In the RANK literature, the natural rate is typically defined as the notional real interest rate that would arise in an economy where all prices were flexible. Instead, our definition of the natural rate follows Keynes (1936) and is defined as the real interest rate consistent with output remaining constant at some particular level. Importantly, while this definition of the natural rate coincides with the real interest rate in the flexible price limit in RANK economies and HANK economies with acyclical risk, the two concepts diverge when risk is countercyclical. See Section 2.3 for a detailed discussion.

flation, pushing the economy into a recession and rendering the initial pessimistic belief self-fulfilling. This is analogous to an "endogenous" negative demand shock.

If monetary policy instead lowers nominal rates sufficiently in response to the lower natural rate, it discourages households from reducing their current spending. This prevents output from declining, preventing the initial beliefs from being confirmed in equilibrium.[3] We show that standard monetary policy rules, even when they satisfy the Taylor principle, do not respond sufficiently strongly to these endogenous fluctuations in the natural rate, and leave the economy susceptible to self-fulfilling non-fundamental fluctuations in output and inflation. Moreover, we provide an exhaustive characterization of the non-fundamental dynamics that these self-fulfilling beliefs can drive.

In order to prevent these non-fundamental fluctuations and to implement a unique equilibrium, monetary policy must adjust the nominal rate one-for-one with these endogenous fluctuations in the natural rate. This is analogous to the RANK literature, where *exogenous* demand shocks can cause the flexible-price real interest rate to fluctuate (Galí, 2015). To optimally neutralize the effects of these exogenous demand shocks on output and inflation, monetary policy should move nominal interest rates one-for-one with changes in the flexible-price real interest rate. In a HANK economy with countercyclical risk, however, the economy is susceptible to "endogenous demand shocks", and monetary policy should also track these *endogenous* changes in the natural rate in order to implement a unique equilibrium. Importantly, while failing to track the flexible-price real interest rate in RANK leads to a *unique* albeit (sub-optimal) equilibrium, failing to track the natural rate in HANK engenders indeterminacy.

It is also important to note that the source of multiplicity described above is conceptually distinct from those identified in the literature on liquidity traps. In particular, Benhabib et al. (2001b) show that the possibility of a binding ELB (effective lower bound) on nominal rates results in multiple equilibria and global indeterminacy in RANK. By contrast our paper purposely abstracts from an ELB, in order to highlight that countercyclical risk is a distinct force driving the global indeterminacy in HANK.

Our analysis goes beyond most existing studies of determinacy in HANK economies by considering *global*, rather than *local* determinacy.[4] The HANK literature has shown that achieving local determinacy of equilibria in HANK economies can be more demanding relative to RANK economies. Acharya and Dogra (2020); Bilbiie (forthcoming); Auclert et al. (2023); Ravn and Sterk (2021) find that if income risk is countercyclical, then the Taylor principle is not sufficient to ensure local determinacy. However, they also show that while the simple Taylor principle fails, a "cyclical-risk augmented Taylor principle", which demands an even stronger response to inflation than in RANK, is sufficient for ensuring *local* determinacy. We show that while a stronger response may be sufficient for local determinacy, it *cannot*

[3] In the same fashion, if households have optimistic beliefs about the economy, they perceive that they will face lower income risk, causing them to reduce their desired level of precautionary savings. This in turn puts upward pressure on the natural interest rate. If monetary policy does not raise the policy rate sufficiently in response to this upward pressure on the natural rate, the higher spending by households leads to higher output and inflation, confirming the initial optimistic beliefs. This acts like a positive endogenous demand shock.

[4] Ravn and Sterk (2021) study a HANK economy with search frictions and also find that global indeterminacy can emerge in their economy with countercyclical income risk. They find that a second "unemployment trap" steady state with 100% unemployment may emerge alongside the targeted equilibrium, implying global indeterminacy in their model. To the best of our knowledge, Ravn and Sterk (2021) is the only other paper which finds global indeterminacy in HANK economies. While our characterization of global indeterminacy in the form of two steady states is complementary to theirs, we also analytically characterize global dynamics that can arise in our economy due to the global indeterminacy. Moreover, we also discuss what policies can eliminate this indeterminacy.

rule out global indeterminacy if risk is even mildly countercyclical.[5] In other words, increasing nominal rates in response to inflation, no matter how aggressively, is not sufficient to implement a unique equilibrium. Our analysis also suggests that conclusions drawn about equilibrium uniqueness based on local stability analysis can be misleading, and tend to understate the prevalence of multiple equilibria in HANK economies in the form of multiple steady states, as well as in the form of periodic cycles.

Our paper also contributes to the literature which studies equilibrium determinacy in heterogeneous agent economies with no nominal rigidities. Kaplan et al. (2023) study multiplicity of equilibria in a flexible-price heterogeneous-agent incomplete-market economy with nominal government debt. They shows that two steady-states exist in their heterogeneous agent economy economy and that the price level and inflation are not uniquely determined. Bassetto and Cui (2018) and Farmer and Zabczyk (2019) study overlapping generations economies and show that price-level indeterminacy may emerge. Brunnermeier et al. (2020) and Miao and Su (forthcoming) study fiscal rules that can deliver price level determinacy in incomplete markets economies in which agents face rate-of-return risk. In contrast, we study equilibrium determinacy in heterogeneous agent economies with nominal rigidities.

Our paper is also related to the older literature which studied global determinacy in RANK. Benhabib et al. (2001a) study the global determinacy properties of RANK economies under standard monetary policy and fiscal rules. Their findings imply that the Taylor principle delivers global determinacy unless (i) households enjoy utility from holding money, and the cross-partial derivative of the utility function $\frac{\partial^2 u(c,m)}{\partial c \partial m} < 0$, or (ii) if money is an input in the production function. Benhabib and Eusepi (2005) study a RANK economy with physical capital, and show that global indeterminacy in the form of periodic cycles around the targeted steady state may emerge.

Finally, Beaudry et al. (2020) argue that alongside business cycle shocks, a substantial part of business cycle fluctuations can be explained by deterministic boom-bust cycles. As in our paper, they show that their New Keynesian model with financial frictions and countercyclical risk-premium features deterministic limit cycles via a Hopf-bifurcation, and that this provides a good description of U.S. business cycle data. Instead, our focus is on understanding how countercyclical income risk in a HANK economy can lead to global indeterminacy.

The rest of the paper is organized as follows. Section 1 describes the model environment, while Section 2 characterizes equilibrium. Section 3 studies local and global determinacy under a standard inflation targeting rule, while Section 4.1 augments the basic inflation targeting rule, allowing the policy rate to also respond to output-gap fluctuations, while Section 4.2 adds inertial behavior to the policy rule. These sections show that while adding these features makes local determinacy of the target equilibrium more likely, these rules also cannot guarantee global determinacy. Section 5 identifies the root of global indeterminacy and proposes a monetary policy rule which implements a unique equilibrium, and Section 6 concludes.

[5] In section 4.1 and 4.2, we also show that, as in RANK, augmenting the standard inflation targeting rule to also respond to output-gap fluctuations, or by adding inertial behavior, makes it easier to attain local determinacy in our HANK economy with countercyclical risk. However, even with these changes, global determinacy still remains elusive.

1 Model

As is well known, HANK models are typically not analytically tractable because the distribution of wealth is a state variable which evolves endogenously. To make our point in the clearest possible way, we use an analytically tractable HANK model in continuous time. In particular, we achieve analytical tractability in our HANK model by assuming that utility is quasi-linear. This ensures that we can characterize the aggregate dynamics of output and inflation, independent of the dynamics of the distribution of wealth.[6] For simplicity, we abstract from aggregate risk.

1.1 Households

There is a continuum of households indexed by $j \in [0,1]$. Each household has identical preferences and the expected discounted lifetime utility of household j at date t can be written as

$$V_j(t) = \max_{\{c_{j,\tau}, n_{j,\tau}\}_{\tau=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau-t)} \left\{ \frac{c_{j,\tau}^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_{j,\tau} \right\} d\tau,$$

where $c_{j,\tau}$ and $n_{j,\tau}$ denote the household's date τ consumption and hours worked respectively. γ measures the elasticity of intertemporal substitution. The household's choices at all dates must satisfy the budget constraint and borrowing constraint:

$$\frac{da_{j,t}}{dt} = (i_t - \pi_t)a_{j,t} + w_t \xi_{j,t} n_{j,t} + D_t - c_{j,t} \qquad \text{with} \qquad a_{j,t} \geq -\underline{a} \tag{1}$$

Households can trade a short-term risk-free nominal bond with return i_t. The real value of bond holdings of household j at time t are denoted by $a_{j,t}$. Household j supplies $\xi_{j,t} n_{j,t}$ effective labor hours and earns labor income $w_t \xi_{j,t} n_{j,t}$, where w_t denotes the real wage. Since idiosyncratic productivity $\xi_{j,t}$ is stochastic, households face labor income risk. In particular, we assume that ξ_j follows a 2-state Poisson process, $\xi_j \in \{\xi_l, \xi_h\}$, where $\xi_h > \xi_l$.[7] In addition to labor income, each household also receives dividends from firms. For simplicity, we assume that all households receive an equal share of dividends D_t. P_t denotes the aggregate price level in the economy.[8] In addition, each household faces a borrowing constraint, which states that their wealth cannot fall below $-\underline{a}$, where $\underline{a} \geq 0$.[9]

$\lambda_{l,t}$ denotes the rate at which a household with productivity ξ_h switches to productivity ξ_l, while $\lambda_{h,t}$ denotes the rate at which a household with productivity ξ_l switches to a ξ_h at date t. We allow the switching intensities to vary over time to capture the notion that households face different levels of income risk during economic expansions and contractions. In particular, we assume that

$$\lambda_{l,t} = \overline{\lambda}_l y_t^{-\Theta}, \tag{2}$$

[6] Lagos and Wright (2005) also use quasi-linear preferences for tractability in the context of their monetary-search model.

[7] It is trivial to allow for a $n-$state Markov process with $n > 2$, and does not affect the analytical tractability of the model.

[8] Since we work with the cashless limit of the economy, the short-term nominal bonds are the unit of account in the economy and $P(t)$ denotes the price of output in terms of the bond.

[9] $\underline{a}$ can be set to zero without loss of generality. In fact, as we show later, the exact value of $\underline{a}$ does not affect the dynamics of aggregate output and inflation.

where Θ controls the *cyclicality of income risk*. $\Theta > 0$ implies that when output is above its steady state level[10] (an expansion), ξ_h households are less likely to transition to the low productivity state. Taking some artistic liberty and treating the ξ_l state as "unemployment", the specification above would imply that households face a lower risk of becoming unemployed during economic expansions. In contrast, $\Theta = 0$ corresponds to *acyclical income risk*: the probability of transitioning from the ξ_h to ξ_l state is independent of the level of economic activity.[11]

Since the rate at which households transition from productivity ξ_h to ξ_l depends on y_t, the fraction of households with productivity ξ_l (given by η_t) and with productivity ξ_h (given by $1 - \eta_t$) would change with the level of economic activity if λ_h was constant. To avoid this complication, we assume that the rate $\lambda_{h,t}$ adjusts to ensure $\eta_t = \eta$ for all dates t.[12]

1.2 Firms

There is a continuum of monopolistically competitive firms indexed by $k \in [0, 1]$. At any date t, firm k produces a differentiated intermediate good $y_{k,t}$, which it sells to a representative final-goods firm. The final-goods firm combines the varieties using a CES aggregator to produce the final-good y_t:

$$y_t = \left[\int_0^1 y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}} dk \right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{3}$$

which yields the standard demand system for each variety k:

$$y_{k,t} = \left(\frac{P_{k,t}}{P_t} \right)^{-\varepsilon} y_t \tag{4}$$

Each intermediate goods producer uses a linear production function $y_k(t) = \ell_{k,t}$, where $\ell_{k,t}$ denotes the effective units of labor employed by firm k. Then, the period t profit of firm k can be written as:

$$D_{k,t} = \frac{P_{k,t}}{P_t} y_{k,t} - (1 - \tau) w_t y_{k,t} - T_{k,t},$$

where τ denotes a payroll subsidy, which we set to $\tau = \varepsilon^{-1}$ to eliminate the average monopolistic markup. Furthermore, this payroll subsidy is financed by imposing a lumpsum tax $T_{k,t} = \tau w_t y_{k,t}$, which the firm treats as given. Then, in symmetric equilibrium, $y_{k,t} = y_t$ and dividends of any firm k can be written as $D_{k,t} = D_t = (1 - w_t) y_t$ for all $k \in [0, 1]$. Finally, nominal rigidities are captured by a standard forward-looking Phillips curve:

$$\dot{\pi}_t = \rho (\pi_t - \pi^\star) - \kappa (w_t - 1), \tag{5}$$

[10]The steady state level of output in the targeted steady state is normalized to 1. See Appendix A.3 for details.

[11]While countercyclical income risk is arguably the empirically relevant benchmark , there is no consensus on the exact measure of countercyclicality. For example, Storesletten et al. (2004) find that the standard deviation of persistent shocks to log household income increases from 0.12 to 0.21 as the aggregate economy moves from peak to trough. However, Guvenen et al. (2014) finds that while the variance of income is acyclical, the left-skewness is countercyclical. More recently, Nakajima and Smirnyagin (2019) find that using a broader definition of income reconciles the seemingly contradictory findings of Storesletten et al. (2004) and Guvenen et al. (2014); they find that both the variance and left-skewness is countercyclical. In any case, our formulation is consistent with both the variance and left-skewness being countercyclical.

[12]The details are available in Appendix A.2.

where $\pi^\star$ is the inflation target.[13] $w_t - 1$ denotes the deviation of marginal cost from the flexible-price benchmark,[14] and $\kappa > 0$ captures the slope of the Phillips curve.

1.3 Monetary and Fiscal Policy

Monetary Policy Monetary policy controls the nominal rate i_t, setting it according to a standard interest rate rule. While we later consider a host of monetary policy rules, in our baseline model, we specify monetary policy as a simple inflation-targeting rule:

$$i_t = \bar{r} + \pi^\star + \phi_\pi \left(\pi_t - \pi^\star \right) \qquad \text{where} \qquad \phi_\pi > 1, \tag{6}$$

$\phi_\pi > 1$ in (6) denotes how aggressively the central bank raises the nominal rate when inflation is above target. As is standard in the RANK literature, we set the intercept to $\bar{r}$, which denotes the real interest rate in the flexible-price limit of our economy.[15] As Appendix A.4 shows, since we abstract from aggregate shocks, the unique equilibrium in the flexible-price limit features a constant level of output $y_t = 1$ and real interest rate $\bar{r}$ at each date t.

Importantly, policy rule (6) does not impose an effective-lower bound (ELB) on nominal rates. We purposely make this choice to make it clear that the source of multiplicity that we uncover is conceptually different from that in Benhabib et al. (2001b), where multiple equilibria arise due to the presence of an ELB.

Fiscal policy For simplicity, we set government expenditures to zero, and assume that the government runs a balanced-budget at each date: $\tau w_t y_t = T_t$, where $\tau w_t y_t$ is the payroll subsidy paid out to firms, and T_t is the lumpsum tax on the same firms.

2 Equilibrium

2.1 Household decisions

Since households have quasi-linear preferences, the date t optimal consumption decision of household j with idiosyncratic productivity ξ_j and wealth a_j can be written as:

$$c_t(a_j, \xi_j) = \left(\frac{\xi_j w_t}{\psi} \right)^\gamma \tag{7}$$

Equation (7) shows that the optimal consumption of household j does not depend on their wealth. Consequently, in equilibrium, all households with idiosyncratic productivity ξ_j, $j \in \{l, h\}$ enjoy the same level of consumption. We refer to the date t consumption of a household with productivity ξ_j,

[13] $\pi^\star$ can be set to 0 without loss of generality.
[14] $w_t = 1$ for all t in the the flexible-price benchmark.
[15] See Appendix A.4 for details on the flexible price limit of our model.

$j \in \{l, h\}$ as $c_{j,t}$. Normalizing $\psi = \left[(1-\eta)\xi_h^\gamma + \eta\xi_l^\gamma\right]^{\frac{1}{\gamma}}$, we have:

$$c_{h,t} = \frac{\xi_h^\gamma}{(1-\eta)\xi_h^\gamma + \eta\xi_l^\gamma} w_t^\gamma \qquad \text{and} \qquad c_{l,t} = \frac{\xi_l^\gamma}{(1-\overline{\eta})\xi_h^\gamma + \overline{\eta}\xi_l^\gamma} w_t^\gamma, \tag{8}$$

where $c_{h,t} > c_{l,t}$. While the consumption of each household does not depend on their wealth, the amount of leisure they enjoy does depend on how wealthy they are. Comparing two households with the same idiosyncratic productivity, the household with higher wealth tends to enjoy more leisure. Furthermore, Appendix A.1 shows that for a given real interest rate, the expected consumption growth of households with productivity ξ_l is always greater than that of households with productivity ξ_h. Thus, in equilibrium, all households with productivity ξ_l are borrowing constrained. In contrast, ξ_h households are *on* their Euler equation, which can be written as:

$$\frac{\dot{c}_{h,t}}{c_{h,t}} = \underbrace{\gamma\Big(i_t - \pi_t - \rho\Big)}_{\text{intertemporal-substitution}} \underbrace{+\gamma\lambda_{l,t}\left[\left(\frac{c_{l,t}}{c_{h,t}}\right)^{-\frac{1}{\gamma}} - 1\right]}_{\text{precautionary savings}} \tag{9}$$

The first term on the RHS of (9) shows that the consumption growth of unconstrained households depends positively on the real interest rate $r_t = i_t - \pi_t$: the *intertemporal-substitution* channel. A higher real interest rate, holding all else constant, incentivizes the unconstrained households to delay consumption, raising their consumption growth. The second term on the RHS of (9) captures the *precautionary-savings* motive. The larger the drop in consumption when a ξ_h households transitions to the ξ_l state (smaller c_l/c_h), the stronger is this motive. Similarly, holding c_l/c_h fixed, a higher risk aversion (smaller γ) or a higher probability of transitioning from ξ_h to ξ_l (higher $\lambda_{l,t}$) also strengthen this motive and increase consumption growth for any given real interest rate.

2.2 Market Clearing

Goods market clearing requires that the amount of final good produced must be consumed:

$$y_t = (1-\eta)\, c_{h,t} + \eta c_{l,t}, \tag{10}$$

where the RHS of (10) is equal to aggregate consumption. Using (8), Appendix A.3 shows that (10) implies a log-linear relationship between aggregate output and wages:

$$w_t = y_t^{\frac{1}{\gamma}} \tag{11}$$

Since wages $w = 1$ in the targeted steady state, (11) also implies that the steady state level of output in the targeted steady state is $y = 1$. Using (8) and (11) also implies that $c_{h,t}$ and $c_{l,t}$ can be rewritten as:

$$c_{h,t} = \frac{\xi_h^\gamma}{(1-\eta)\xi_h^\gamma + \eta\xi_l^\gamma} y_t \qquad \text{and} \qquad c_{l,t} = \frac{\xi_l^\gamma}{(1-\eta)\xi_h^\gamma + \eta\xi_l^\gamma} y_t \qquad \Rightarrow \qquad \frac{c_{h,t}}{c_{l,t}} = \left(\frac{\xi_h}{\xi_l}\right)^\gamma, \tag{12}$$

While consumption of both ξ_h and ξ_l households co-moves with output, the ratio $c_{h,t}/c_{l,t} > 1$ is constant over time. Finally, at any date t, since all ξ_l households are borrowing constrained and have $a = -\underline{a}$, asset market clearing requires that asset holdings of ξ_h households as a whole is given by $\frac{\eta}{1-\eta}\underline{a}$.

2.3 Aggregate dynamics, the neutral rate and the natural rate of interest

As is standard in the textbook treatment of the RANK model, the aggregate dynamics of output and inflation in our tractable HANK economy can be also summarized by an IS curve and a Phillips curve. Using (11) in (5), we can express the Phillips curve in terms of output and inflation:

$$\dot{\pi}_t \quad = \quad \rho\left(\pi_t - \pi^\star\right) - \kappa\left(y_t^{\frac{1}{\gamma}} - 1\right) \tag{13}$$

Appendix A.4 shows that using (12) and (2) in (9), the "IS curve" in our HANK economy is given by:[16]

$$\frac{\dot{y}_t}{y_t} \quad = \quad \gamma\left(i_t - \pi_t - r^\star(y_t)\right), \tag{14}$$

As in the textbook 3-equation RANK model, (14) shows that output growth is positive when the monetary policy implements a real interest rate $r_t = i_t - \pi_t$ which is higher than *natural rate of interest* $r^\star(y_t)$. Here, we define $r^\star(y)$ as the real interest rate which sets $\dot{y}_t/y_t = 0$ and $y_t = y$ for all t, i.e., $r^\star(y)$ is the real interest rate which is consistent with output remaining constant at $y_t = y$ for all t.

In the RANK limit of our model ($\overline{\lambda}_l = 0$), since households do not face consumption risk and because we abstract from aggregate shocks, the natural rate is constant over time and simply equal to the discount rate: $r^\star(y) = \rho$. This means that setting $r = r^\star = \rho$ in RANK is consistent with output remaining fixed at any level of output y. The *natural* rate $r^\star(y)$ also coincides with the real interest rate in the flexible-price limit of our economy, $\overline{r} = r^\star(y) = \rho$.

The same is true in HANK provided that risk is acyclical ($\Theta = 0$). The only difference is that in HANK with $\Theta = 0$, both the natural rate and the flexible-price real interest rate are lower than in RANK, and are instead given by $r^\star(y) = \overline{r} = \rho - \sigma$, where $\sigma = \overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right)$ measures the expected increase in marginal utility when a ξ_h household transitions to lower idiosyncratic productivity ξ_l, and captures the fact that households face consumption risk.[17] Since households face consumption risk, they save for precautionary reasons, and this requires a lower interest rate for the asset market to clear.

However, if risk is countercyclical ($\Theta > 0$), the natural rate is *endogenous* and co-moves with output:

$$r^\star(y) = \rho - \sigma y^{-\Theta} \qquad \text{with} \qquad \frac{dr^\star(y)}{dy} = \sigma\Theta y^{-(1+\Theta)} > 0, \tag{15}$$

i.e., the real interest rate consistent with output being constant at a particular level y, now depends on the level of output itself. Moreover, the natural rate $r^\star(y)$ is an increasing function of y. Thus, in our

[16]In equilibrium, while the actual value of the debt limit $\underline{a}$ does not affect aggregate dynamics of y_t and π_t, it does matter for the wealth distribution that emerges in equilibrium. However, quasi-linear preferences render our economy block-recursive, allowing us to characterize the dynamics of output and inflation independently of the wealth distribution.

[17]Clearly, $\sigma = 0$ if the probability of transitioning to the low productivity state is zero ($\overline{\lambda}_l = 0$), or if the consumption across the two idiosyncratic productivity levels is the same (which occurs if $\xi_h = \xi_l$). In all other cases, when a transition to the ξ_l state results in a decline in consumption, $\sigma > 0$.

HANK economy with countercyclical risk, the flexible price real interest rate $\bar{r} = \rho - \sigma$ is, in general, different than the natural rate $r^{\star}(y)$. The two concepts only coincide in steady state when $y = 1$: $\bar{r} = r^{\star}(1)$. To see why, recall that if output was lower than $y = 1$, then since risk is countercyclical, households face greater consumption risk. Given all else, this causes them to reduce their current consumption demand and increase their precautionary savings. This greater desire to save implies that a lower real interest rate is required to clear asset markets and keep demand constant at that lower level.[18]

Notice that our definition of the natural rate of interest $r^{\star}(y)$ differs from how the term "natural rate" is used in the New Keynesian literature (see, e.g., Woodford (2003a); Galí (2015)). In the New Keynesian literature, the natural rate is typically defined as the real interest rate which would prevail in the flexible-price limit of the economy, which is equal to $\bar{r} = r^{\star}(1) = \rho - \sigma$ in our HANK model. This flexible-price real interest rate depends on exogenous parameters, and potentially varies over time only in response to *exogenous* shocks, e.g. shocks to the discount rate ρ. Importantly, it does not depend on *endogenous* variables such as the level of output. Consequently, in RANK and in HANK with acyclical risk, our definition of natural rate coincides with the flexible-price real interest rate. In contrast, when risk is countercyclical $\Theta > 0$, the natural rate $r^{\star}(y)$, as we define it, *does* depend on *endogenous* variables, specifically output: in a weak economy, where output is below its flexible price level, a lower real interest rate is required to maintain demand, and hence output, at that level. In contrast, even in our economy with $\Theta > 0$, the flexible-price real interest rate does not depend on endogenous variables, $\bar{r} = \rho - \sigma$. Thus, in our HANK economy with countercyclical risk our definition of the natural rate does not always coincide with the flexible-price real interest rate: there are many *natural* rates $r^{\star}(y)$, one for a given level of y, but there is a unique flexible-price real interest rate $\bar{r}$, which coincides with the natural rate consistent with $y = 1$. Our choice of terminology hearkens back to Keynes (1936) (pp. 242-243):

> *For every rate of interest there is a level of employment for which that rate is the "**natural**" rate, in the sense that the system will be in equilibrium with that rate of interest and that level of employment. ... we might term the **neutral** rate of interest, ... the natural rate in the above sense which is consistent with full employment, given the other parameters of the system.* [emphasis ours]

Thus, following Keynes (1936), we also refer to the flexible-price real interest rate as the *neutral rate*, which is not the same as the *natural* rate in our HANK economy with countercyclical risk.

2.4 Steady states: an IS-LM representation

Our model admits an IS-LM representation in steady state, which is useful to explain how countercyclical risk affects the steady state properties of the economy. The fact that the natural rate $r^{\star}(y)$ depends on the level of y in our HANK economy with countercyclical risk can be visualized in terms of an upward-sloping *long-run IS curve*, which is defined by the locus of (y, r) which sets $\dot{y} = 0$ in (14),

[18] If output was higher than $y = 1$, then households would face lower consumption risk. This causes them to reduce their precautionary savings, and thus a higher real interest rate is needed to keep demand constant at that higher level.

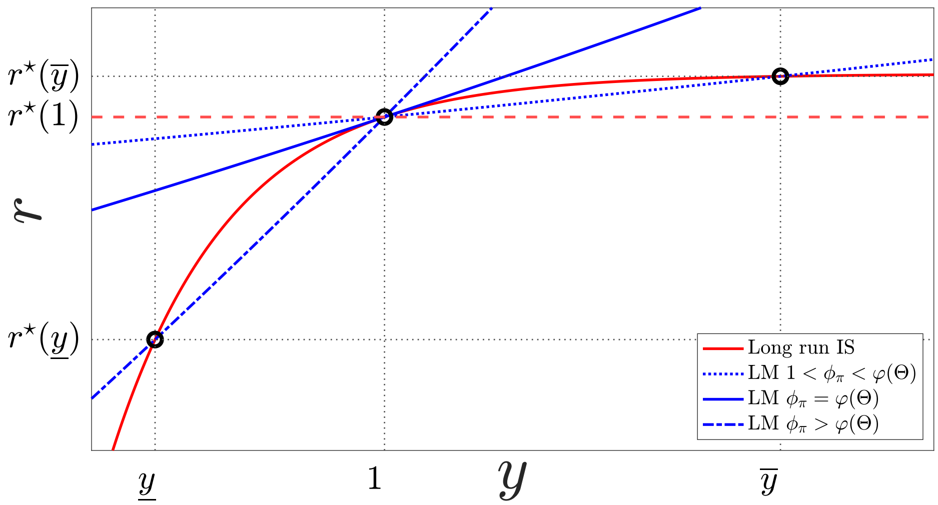


Figure 1: **Multiple steady states with countercyclical risk $\Theta > 0$.** The solid-red curve depicts the long-run IS curve in HANK with countercyclical risk, while the dashed-red curve depicts the long-run IS curve in the acyclical risk ($\Theta = 0$) benchmark. The solid-, dotted- and dot-dashed blue curves depicts the LM curves for different values of ϕ_π. $y = 1$ and $r = r^\star(1)$ denotes the targeted steady state.

and can be written as:[19]

$$r = \rho - \sigma y^{-\Theta} \tag{16}$$

This is depicted by the solid-red curve in Figure 1. In contrast, in RANK or in HANK with acyclical risk, the long-run IS curve is horizontal. The long-run IS curve in HANK with acyclical risk is depicted by the dashed-red horizontal line at $r = r^\star(1)$ in Figure 1.[20]

The LM curve (depicted by the solid-blue curves in Figure 1) can be derived by setting $\dot{\pi} = 0$ in (13), and combining this expression with the monetary policy rule (6):[21]

$$r = \overline{r} + \frac{\kappa(\phi_\pi - 1)}{\rho}\left(y^{\frac{1}{\gamma}} - 1\right) \tag{17}$$

Any intersection of the long-run IS curve (16) and LM curve (17) constitutes a steady-state. Clearly, $y = 1$ and $r = \overline{r} = r^\star(1)$ satisfy both (16) and (17), implying that there always exists one steady state in which output equals its flexible-price level $y = 1$ and inflation is on target $\pi = \pi^\star$. Consequently, the steady state real interest rate in this steady state is equal to the flexible-price real interest rate $\overline{r} = r^\star(1)$. We will refer to this as the *targeted steady state*. However, since both the IS curve and LM are both strictly increasing relationships between the steady state real interest rate r and steady state output y, there is a possibility that they intersect more than once depending on the relative slopes of the two curves. In fact, in general, there are two steady states, one of which is the always the targeted steady state. As Appendix B.3 shows, for a given $\Theta > 0$, unless $\phi_\pi = \varphi(\Theta) \equiv 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$, a second *untargeted* steady state exists. Only in the knife edge case $\phi_\pi = \varphi(\Theta)$, is the targeted steady state unique.

[19] Note that this is different from traditional old-Keynesian models in which the IS curve is downward-sloping in (y, r) space. To obtain a downward sloping IS curve in a micro-founded HANK model, one would need risk to be *procyclical*.

[20] In the RANK limit, this horizontal line is simply at a higher level since the neutral rate is higher in RANK.

[21] The LM curve is an upward sloping relationship between r and y since we have assumed that $\phi_\pi > 1$.

The emergence of the second steady state is rooted in the fact that, with countercyclical risk, households face greater uncertainty when output is lower. To see how this can result in multiple steady states, consider the case in which $1 < \phi_\pi < \varphi(\Theta)$. In this case, the LM curve is relatively flat and the untargeted steady state features a higher level of output $\overline{y} > 1$ and inflation above target $\overline{\pi}^g > 0$, as well as a real interest rate higher than the neutral rate. To see why such a steady state can arise, suppose that households believe that the economy will have higher output $\overline{y} > 1$ forever. Because risk is countercyclical, this belief regarding higher output implies that households perceive that they face lower income risk. This causes them to lower their desired level of precautionary savings and to increase consumption spending. Owing to the presence of nominal rigidities, this higher spending puts upward pressure on output and inflation. When $1 < \phi_\pi < \varphi(\Theta)$, the monetary monetary policy rule (6) raises nominal rates in response to the higher inflation, but the implied increase in real interest rates is not large enough to dissipate this higher demand, thus allowing the initial beliefs of higher output to become self-fulfilling. Finally, since households face less risk and save less in this steady state, the lower stock of savings implies that a higher real interest rate $r = r^\star(\overline{y}) > r^\star(1)$ clears asset markets.

Increasing ϕ_π towards $\varphi(\Theta)$ induces a larger increase in nominal rates in response to the same increase in inflation, implying that monetary policy raises real interest rates more. These higher real rates lower output in the second steady state, bringing $\overline{y}$ closer to 1 (graphically, this makes the blue-dotted LM curve steeper). In fact, as one increases ϕ_π all the way to $\varphi(\Theta)$, the dotted-blue LM curve now becomes the solid-blue LM curve in Figure 1, which is tangent to the long-run IS curve at the targeted steady state. In this knife-edge case, the only steady state is the targeted steady state $y = 1, \pi = \pi^\star$, implying that the real interest rate is equal to $r^\star(1)$.

Increasing ϕ_π even further ($\phi_\pi > \varphi(\Theta)$), makes the LM curve even steeper (dot-dashed blue curve in Figure 1), and multiple steady states emerge again. However, the untargeted steady state now features lower output $\underline{y} < 1$ and below target inflation $\underline{\pi} < \pi^\star$. With countercyclical risk, lower output ($\underline{y} < 1$) implies that households face more risk in this steady state compared to the targeted steady state, prompting them to increase their precautionary savings. This causes them to lower spending, which puts downward pressure on output and inflation. Monetary policy, following the rule (6), lowers the nominal rate in response to the lower inflation, but despite this, the equilibrium real interest rate is too high to discourage the higher precautionary savings. This reinforces lower household demand, trapping the economy at a lower level of economic activity $y = \underline{y} < 1$. Furthermore, the stronger desire to save requires a lower real interest rate to clear asset markets in this steady state, $r = r^\star(\underline{y}) < r^\star(1)$. Appendix B.3 shows that this untargeted steady state persists even if we keep raising ϕ_π further (as long as it remains finite). Increasing ϕ_π further only has the effect of making output in the untargeted steady state even lower.

While Figure 1 and the discussion above show that the untargeted steady state can feature higher or lower output than in the targeted steady state, in what follows, we will focus on the scenario in which the targeted steady state is *locally determinate*. As we show in Section 3, this requires that $\phi_\pi > \varphi(\Theta)$, implying that the untargeted steady state always exists and features lower output and inflation than in the targeted steady state. FInally, it is useful to note that if risk is acyclical ($\Theta = 0$), the long-run IS curve is horizontal and only intersects the LM curve once, implying that countercyclical risk is key to the existence of the non-targeted steady state.

2.5 Aggregate dynamics

Given the monetary policy rule (6), Proposition 1 shows that the aggregate dynamics of our HANK economy are described by the 2-dimensional system of ordinary differential equations (ODEs). However, instead of working directly with output and inflation, it is more convenient to describe the dynamics of the (scaled) *output-gap*, and the *inflation-gap*. We define the output-gap as the log-deviation of output from its flexible-price level scaled by γ^{-1}: $x = \gamma^{-1}(\ln y - \ln 1)$, and the inflation gap as the level-deviation of inflation from its target $\pi^g = \pi - \pi^\star$.

Proposition 1. *Given the interest rate rule (6), the aggregate dynamics of the economy can be written as:*

$$\dot{x}_t = (\phi_\pi - 1)\,\pi^g_t - \left(r^\star(x_t) - \bar{r}\right) \tag{18a}$$

$$\dot{\pi}^g_t = \rho\pi^g_t - \kappa\left(e^{x_t} - 1\right) \tag{18b}$$

where $r^\star(x_t) - \bar{r} = \sigma\left(1 - e^{-\gamma\Theta x_t}\right)$ denotes the difference between the natural rate of interest and neutral rate of interest. The target steady state is given by $x = \pi^g = 0$.

Proof. See Appendix A.4. □

Equations (18a)-(18b) nest the RANK benchmark: in RANK $\sigma = 0$ and so, $r^\star(x_t) = \bar{r} = \rho$ for any x, implying that the last two terms on the RHS of (18a) eliminate each other. The same is true in HANK if risk is acyclical ($\Theta = 0$), even through the natural and neutral rates of interest are both lower in this case than in RANK. Thus, (18a) and (18b) show that the global dynamics in the RANK benchmark and in a HANK economy with acyclical ($\Theta = 0$) are identical as long as ϕ_π is the same in both economies. Thus, simply the presence of risk does not necessarily alter the dynamics of output and inflation. In contrast, when risk is countercyclical, (18a) reveals that an extra force shapes global dynamics relative to the RANK and acyclical risk benchmark. In particular, fluctuations in the output-gap drive an *endogenous* gap between the natural rate and neutral rate of interest, which in turn feeds back in to the dynamics of output and inflation.

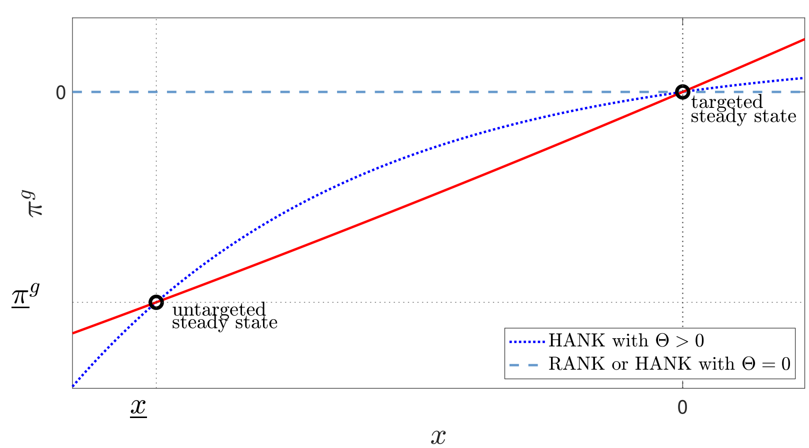


Figure 2: **Phase space:** The solid-red curve depicts the $\dot{\pi}^g_t = 0$ nullcline, while the dotted-blue curve depicts the $\dot{x} = 0$ nullcline in HANK with countercyclical risk. In the RANK benchmark or if risk is acyclical, the $\dot{x} = 0$ nullcline is depicted by the dashed horizontal line.

Figure 2 graphically depicts the nullclines associated with (18a)-(18b) in (x, π^g) space. The solid upward sloping line represents the $\dot{\pi}_t^g = 0$ nullcline and is unaffected by the presence of risk or the cyclicality of risk. In contrast, in the RANK benchmark $\sigma = 0$) (or in HANK with $\Theta = 0$), the $\dot{x} = 0$-nullcline is depicted by the dashed-horizontal curve at $\pi^g = 0$, while in HANK with $\Theta > 0$, it is depicted by the dotted-blue curve. The reason that the nullcline is flat in RANK (or in HANK with $\Theta = 0$) is that $r^\star(x)$ is constant, but when risk is countercyclical, $r^\star(x)$ co-moves with output, and is thus upward sloping. Importantly, as discussed earlier, whenever $\phi_\pi > \varphi(\Theta)$, there are always two steady states (depicted by two intersections of the dotted-blue and solid-red curves). In contrast, in the RANK benchmark or when risk is acyclical, the curves only intersect once.

Parameterization While our characterization of global determinacy is analytic, we parameterize the model when we plot equilibrium trajectories. In our preferred parameterization, we set the discount rate ρ to be consistent with a real interest rate of 4% in steady state. We set the coefficient of relative risk aversion $\gamma^{-1} = 2$ (which is equivalent to an IES, $\gamma = 0.5$). We set the rate at which ξ_h households transition to ξ_l productivity $\overline{\lambda}_l = 0.013$ based on the estimates of Bilbiie et al. (2023), who estimate a model which is very similar to ours (which is formulated in discrete time).[22] Translating the estimates of Bilbiie et al. (2023) into continuous time also yields a range for Θ between 21.98 and 29.9, with the modal estimate of 28.1. To calibrate the relative differences in idiosyncratic productivity ξ_h/ξ_l, we use estimates of the decline in consumption when a household transitions from employment to unemployment. In particular, we set $c_h/c_l = 1.1$, which is consistent with the empirical estimates of the decline in consumption when a household experiences involuntary unemployment.[23] Equation (12), then implies that $\xi_h/\xi_l = 1.23$, i.e, the ξ_h households are 23% more productive than the ξ_l households. Finally, as is common in the literature, we set $\phi_\pi = 1.5$ in the monetary policy rule (6).

3 Local vs global determinacy of equilibrium

As is well known, absent an ELB, uniqueness of equilibrium in the standard textbook three-equation New-Keynesian model (Woodford, 2003a; Galí, 2015) with complete markets featuring *Ricardian* fiscal policy is ensured by imposing the Taylor principle. The Taylor principle dictates that monetary policy should raise nominal rates more than one-for-one in response to an increase in inflation. This ensures that the target equilibrium is *locally* determinate. In fact, Appendix B.1 shows that the Taylor principle also ensures that the targeted equilibrium is globally determinate in the RANK limit of our economy.[24]

[22] Bilbiie et al. (2023) postulate the following functional form to describe the evolution of the probability that a unconstrained household stays unconstrained: $\ln s_t = \ln s_0 + s_1 \ln y_t$. Thus, we need to translate the probability $1 - s_t$ into an arrival rate, which can be accomplished using the conversion formula $1 - s_t = 1 - e^{-\lambda_{l,t}}$, which can be simplified to yield $\ln s_t = -\overline{\lambda}_l y_t^{-\Theta}$. Imposing steady state $y_t = 1$, where $s_t = s_0$ and $\lambda_t = \overline{\lambda}_l$, we can set $\overline{\lambda}_l = -\ln s_0$ and $\frac{d \ln \lambda}{d \ln y} = s_1 = -\overline{\lambda}_l \Theta$. Bilbiie et al. (2023) set $s_0 = 0.987$ and estimate s_1 to lie in the range 0.2880 and 0.3920. This implies $\overline{\lambda}_l = 0.0131$ and that $\Theta \in [21.98, 29.9]$.

[23] A 10% decline in consumption is well within the range of empirical estimates, e.g., Cochrane (1991) finds that the consumption growth of households who lost their job was 24-27% lower than households who did not, Ganong and Noel (2019) find that the consumption of households who become unemployed drops by around 11% when unemployment benefits expire, while Gruber (1997) documents that food consumption falls on average by 6.8% after households become unemployed.

[24] Of course, global determinacy would not obtain if we enforced an ELB. As is well known, Benhabib et al. (2001b) show that imposing an ELB introduces multiple bounded trajectories which are consistent with equilibrium, implying that there can be global indeterminacy even when the target equilibrium is locally determinate. As aforementioned, we purposely do

Following the RANK literature, most papers in the HANK literature have focused on characterizing conditions under which the targeted equilibrium is *locally* determinate (see, e.g., Acharya and Dogra (2020); Bilbiie (forthcoming); Ravn and Sterk (2021) and Auclert et al. (2023)).[25] In particular, these papers argue that if risk is countercyclical, the simple Taylor principle ($\phi_\pi > 1$) may not be sufficient to ensure local determinacy. Instead, local determinacy requires that the central bank respond even more aggressively to increases in inflation compared to the RANK benchmark. This property is also true in the context of our model, as Proposition 2 shows.

Proposition 2 (Local determinacy in HANK with countercyclical risk)**.** *The targeted equilibrium of the economy described by* (18a)-(18b) *is locally determinate if ϕ_π satisfies*

$$\phi_\pi > \varphi(\Theta) \qquad \textit{where} \qquad \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}, \tag{19}$$

provided that risk is not too countercyclical $\Theta \in [0, \Theta^\star)$, where $\Theta^\star \equiv \frac{\rho}{\sigma\gamma}$. If $\Theta > \Theta^\star$, then the targeted equilibrium is locally *determinate for any finite ϕ_π, no matter how large it is.*

Proof. See Appendix B.2. □

Equation (19) is the analog of the "cyclical-risk" augmented Taylor principle derived in Acharya and Dogra (2020), Bilbiie (forthcoming), Auclert et al. (2023) in the context of our model. First, notice that (19) simplifies to the standard Taylor principle $\phi_\pi > 1$ in the RANK limit of our model ($\sigma = 0$). When households face risk ($\sigma > 0$), but this risk is acyclical ($\Theta = 0$), (19) shows that the standard Taylor principle is sufficient for local determinacy. However, (19) shows that monetary policy needs to respond more aggressively to changes in inflation, the more countercyclical risk is.

What does local determinacy mean as opposed to global determinacy? Proposition 2 implies that as long as $\Theta < \Theta^*$, and if the monetary policy rule (6) satisfies (19), then the targeted equilibrium is *locally* determinate. Local determinacy implies that any trajectory, other that $(x_t, \pi_t^g) = (0,0)$, that starts in a small neighborhood of the targeted-steady state $(0,0)$ leaves this neighborhood, i.e., all these trajectories do not remain *bounded* inside this neighborhood. The only trajectory that stays bounded in this small neighborhood is the trajectory $(x_t, \pi_t^g) = (0,0)$, implying that $(0,0)$ is the unique *bounded* equilibrium in its neighborhood. In other words, if one limits analysis to rational expectations equilibria in which (x, π^g) remain forever in a small neighborhood of the targeted-steady state $(0,0)$, then the only bounded equilibrium is $(0,0)$. This follows from the fact that in the neighborhood of $(0,0)$, the dynamics of (x, π^g) are described by the 2-dimensional linear system of ODEs:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \end{bmatrix} = \underbrace{A \begin{bmatrix} x_t \\ \pi_t^g \end{bmatrix}}_{\substack{\text{first-order} \\ \text{terms}}} + \mathcal{O}(x^2) \qquad \text{for} \qquad (x, \pi^g) \to (0,0) \qquad \text{with} \qquad A = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}, \tag{20}$$

and Appendix B.2 shows that the stability of the system is governed by the eigenvalues of the matrix A. (19) ensures that both eigenvalues are *explosive*, i.e., they have positive real parts. Consequently, the

not impose an ELB to highlight that multiplicity of equilibria can emerge in HANK models even absent the ELB.

[25] As mentioned earlier, the one exception is Ravn and Sterk (2021), who also study global determinacy.

only bounded trajectory that remains in the local neighborhood of the steady state $(0, 0)$ is that steady state itself. If we instead start at some point other than $(0, 0)$ in the neighborhood of the targeted steady state, the two explosive eigenvalues imply that the trajectories of (x_t, π_t^g) grow unbounded and leave the neighborhood of $(0, 0)$, which disqualifies the trajectories as valid equilibria. In other words, local determinacy implies that the targeted steady state $(0, 0)$ is *unstable*. In contrast, if the cyclical-risk augmented Taylor principle (19) is not satisfied,or if risk is highly countercyclical ($\Theta > \Theta^{\star}$), then there are multiple bounded trajectories originating in the neighborhood of $(0, 0)$, which converge to $(0, 0)$, implying that the economy is locally *indeterminate*, i.e., the targeted steady state $(0, 0)$ is *stable*. Note that mathematical *instability* corresponds to *determinacy*. Thus, to show that an equilibrium is determinate, we will show that the system is unstable, and that there is only one stable trajectory.

Proposition 2 suggests that as long as risk is not too countercyclical $\Theta < \Theta^{*}$ a large enough ϕ_{π} which satisfies (19) ensures *local* determinacy. Appendix B.1 shows that (19) is also sufficient to guarantee global determinacy in the RANK limit of our economy ($\sigma = 0$), or if risk is acyclical ($\sigma > 0, \Theta = 0$). However, the equilibrium is *not* globally determinate in our HANK economy if risk is even mildly countercyclical (small Θ). This is because as long as $\Theta > 0$, there exists at least one other trajectory other than $(x_t, \pi_t^g) = (0, 0)$ for all t, which remains bounded. This second bounded trajectory is simply the non-targeted steady state $(x_t, \pi_t^g) = (\underline{x}, \underline{\pi}^g)$, which exists if risk is even mildly countercyclical ($\Theta > 0$).[26] This global indeterminacy is not eliminated by raising ϕ_{π}, since the untargeted steady state exists no matter how large ϕ_{π} is, as long as $\Theta > 0$.

The existence of the untargeted steady state implies that, even absent the ELB, our HANK economy with countercyclical risk could *stagnate*, with inflation staying permanently below its target and output being below potential. While this is reminiscent of the secular stagnation literature (e.g. Benigno and Fornaro (2018); Eggertsson et al. (2019)), which showed that the economy can get stuck at low levels of economic activity because monetary policy is stuck at the ELB, our HANK economy stagnates not because of a binding ELB, but because monetary policy fails to fully account for the endogenously lower natural rate. However the hazards induced by the standard Taylor rule in our HANK economy are not limited to this kind of permanent slump. Proposition 3 describes what kind of other non-fundamental fluctuations may plague a HANK economy with countercyclical risk, even if the monetary policy rule satisfies the risk-augmented Taylor principle (19).

Proposition 3 (Global dynamics)**.** *Consider the economy described in Proposition 1 for a given $\Theta > 0$ and assume that (19) is satisfied. Then the global dynamics depend on the magnitude of the cyclicality of risk Θ, and can be split into 3 broad regions.*

1. ***Mildly countercyclical risk** $\Theta \in \left(0, \overline{\Theta}\right)$**:** $\exists \overline{\Theta} > 0$, such that for any $\Theta \in (0, \overline{\Theta})$, there exists a saddle-connection, along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. In this region, the targeted equilibrium is locally determinate, but there is global indeterminacy, as any trajectory starting on the saddle connection also remains bounded forever, while satisfying all equilibrium conditions.*

[26]Given that (19) is satisfied, this untargeted steady state features $\underline{x} < 0$ and $\underline{\pi}^g < 0$. Moreover, there exist even more bounded trajectories since Appendix B.4 shows that as long as (19) is satisfied, the untargeted steady state is locally indeterminate: there is a one-dimensional stable manifold around the untargeted steady state. Any trajectory which originates in this manifold also converges to the untargeted steady state, and thus remains bounded.

2. ***Moderately countercyclical risk*** $\Theta \in [\overline{\Theta}, \Theta^\star]$: *For any* $\Theta \in [\overline{\Theta}, \Theta^\star)$, *trajectories originating in the neighborhood of the targeted steady state* $(0,0)$ *initially diverge away from the steady state but eventually converge to a* super-critical limit-cycle *surrounding* $(0,0)$, *and thus remain bounded and imply the equilibrium is globally indeterminate, even though the targeted steady state is locally determinate. Moreover, the periodicity of the periodic solutions is a decreasing function of* Θ *in the range* $(\overline{\Theta}, \Theta^\star)$.

   In the knife edge case with $\Theta = \overline{\Theta}$, *there exists a* homoclinic orbit, *which is a stable trajectory which connects the untargeted steady state* $(\widetilde{x}, \widetilde{\pi}^g)$ *to itself, and lies on the boundary of the periodic solutions described above. Again, this implies global indeterminacy, even though* $(0,0)$ *is locally determinate.*

   Finally, if $\Theta = \Theta^*$, *the limit-cycles collapse onto the steady state* $(0,0)$. *The equilibrium is globally indeterminate since the higher-order terms ensure that any trajectory starting in the neighborhood of the targeted steady state converges back to* $(0,0)$.

3. ***Highly countercyclical risk*** $\Theta > \Theta^\star$: *For any* $\Theta > \Theta^\star$, *the targeted steady state* $(0,0)$ *is locally indeterminate even if* (19) *is satisfied, i.e., there exists multiple bounded trajectories in the neighborhood of* $(0,0)$, *which converge to the targeted steady state. Since the equilibrium is locally indeterminate, it is also globally indeterminate. In addition to the multiple bounded trajectories which start near the targeted steady state, there also exists a* saddle-connection, *along which, the economy converges to the targeted steady state even if it starts near the untargeted steady state.*

Overall, for any $\Theta > 0$, *i.e, if risk is even mildly countercyclical, there is global indeterminacy. For any finite* ϕ_π, *the inflation targeting rule* (6) *fails at eliminating the existence of multiple bounded equilibria.*

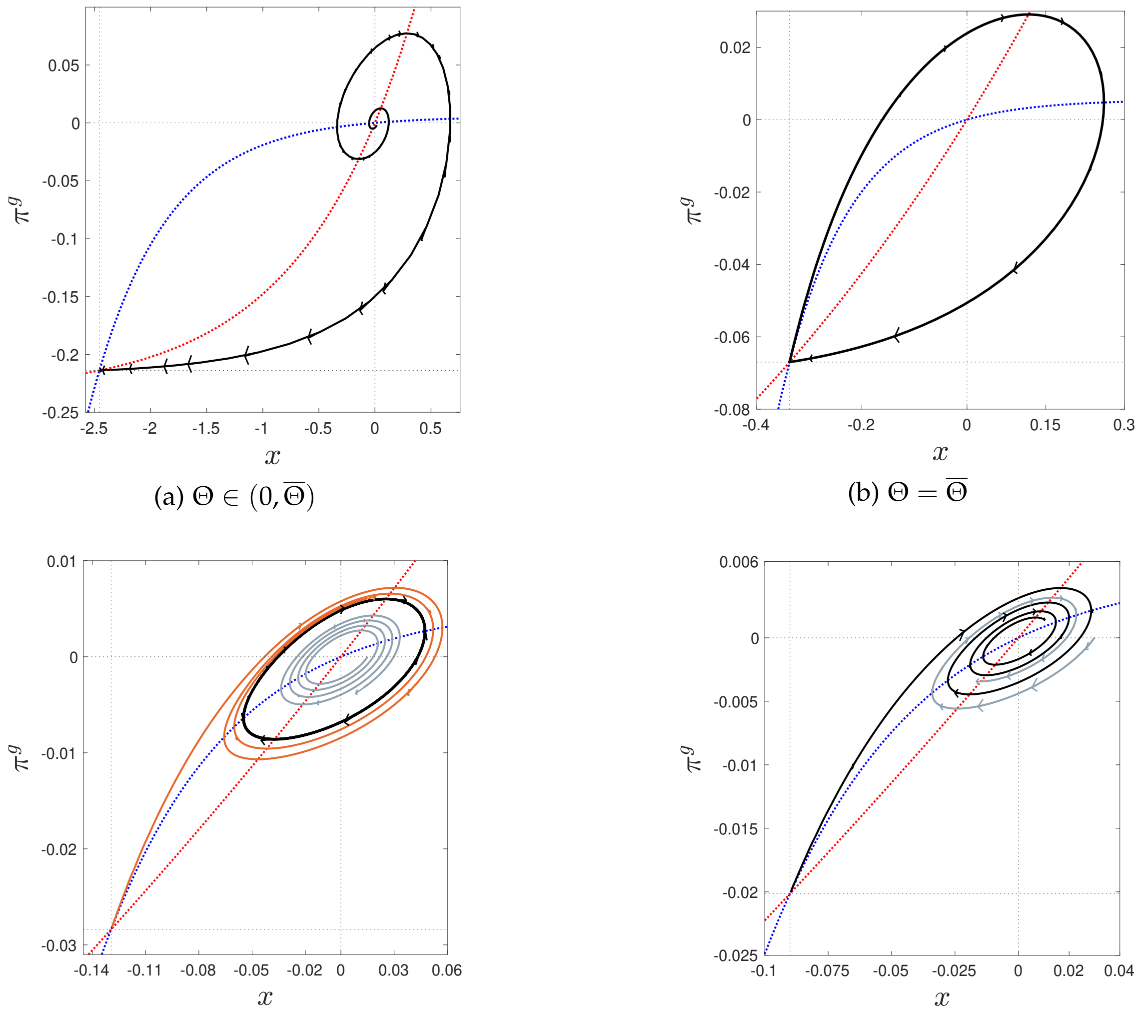
Proof. See Appendix B.5. □

Proposition 3 shows that if risk is even mildly countercyclical (Θ close to 0), the Taylor principle cannot ensure globally determinacy, even though it can deliver local determinacy. The four panels of Figure 3 depict global dynamics of the economy in the regions described in the Proposition above. In all four panels, the dotted blue curve denotes the $\dot{x} = 0$-nullcline and the dotted-dark red curve denotes the $\dot{\pi} = 0$-nullcline. All four panels feature our baseline calibration, but they have different Θ.

To see why the dynamics take this form in each region, it is useful to understand why the HANK economy features global indeterminacy even when the Taylor principle (19) is satisfied, when the RANK economy does not. As Cochrane (2011) explains: absent the ZLB, the Taylor principle ensures a unique bounded equilibrium in the RANK model, because off-equilibrium, "*higher inflation leads the Fed to set interest rates in a way that produces even higher future inflation*". In other words, imposing the Taylor principle induces explosive dynamics if the economy is not on the targeted equilibrium, thus leaving the targeted equilibrium as the *unique bounded equilibrium* in RANK. To see this in our model, imposing the RANK limit $\sigma = 0$ in (18a), the RANK IS curve can be written as:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t^g$$

When the Taylor principle (19) is satisfied, $\phi_\pi - 1 > 0$, and the expression above shows that $\dot{x}_t$ is *increasing* in π_t^g, i.e., any deviation of inflation from its target induces destabilizing dynamics, causing x to change. In fact, the larger is ϕ_π relative to 1, the same deviation in inflation from its targeted value



(a) $\Theta \in (0, \overline{\Theta})$

(b) $\Theta = \overline{\Theta}$

(c) $\Theta \in (\overline{\Theta}, \Theta^{\star})$

(d) $\Theta > \Theta^{\star}$

Figure 3: Global dynamics depending on magnitude of Θ

induces a larger change in x. For example, consider a case in which inflation is below target at date 0, $\pi^g < 0$. Then, if the Taylor principle holds, the IS curve (18a) implies that x_t must decline.[27] Since the Phillips curve (18b) implies that inflation at any date is the net-present discounted value of future marginal costs, π falls further below target over time.[28] These destabilizing dynamics induced by the Taylor principle ensure that any trajectory which originates at any point other than $(0, 0)$ does not remain bounded, and hence is not a valid equilibrium. $(x, \pi^g) = (0, 0)$ is the only *bounded* equilibrium in RANK if the Taylor principle is satisfied.

However, in our HANK economy with countercyclical risk, even if the monetary policy rule satisfies

[27]Output declines following date 0 unambiguously, but whether it declines monotonically over time or not depends on the eigenvalues of the system. While the Taylor principle ensures that the 2 eigenvalues are positive, the eigenvalues can either both be real or both complex with positive real parts. When the eigenvalues are real, x continues to decline monotonically towards $-\infty$, while if the two roots are complex, the trajectory of x is oscillatory with ever increasing amplitude. However, in both cases, as long as the Taylor principle is satisfied, the trajectory for x diverges away from its targeted level of 0 over time.

[28]Again, whether inflation diverges from its targeted level monotonically or in an oscillatory fashion depends on whether output declines monotonically or diverges in an oscillatory fashion, which in turn depends on the eigenvalues of the system. With two real positive roots, inflation also declines monotonically towards $-\infty$. With complex roots, the trajectory of inflation, like output, is also oscillatory but with ever increasing amplitude.

the Taylor principle, the IS equation (18a) (which we replicate below for convenience) now has an extra term $\bar{r} - r^\star(x_t)$:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t^g + \Big(\bar{r} - r^\star(x_t)\Big)$$

While the first term on the right-hand-side (RHS) still induces destabilizing dynamics as in RANK, the second term (which reflects the fact that in HANK with countercyclical risk, $\Theta > 0$, the natural rate $r^\star(x)$ endogenously *co-moves* with x) instead induces *stabilizing* dynamics since $\frac{dr^\star(x)}{dx} = \gamma\Theta e^{-\gamma\Theta x} > 0$. Furthermore, using the fact that $\bar{r} - r^\star(x) = \sigma(e^{-\gamma\Theta x_t} - 1)$, we can use the power-series expansion of the exponential function to rewrite the IS curve as:

$$\dot{x} = \underbrace{(\phi_\pi - 1)\pi^g - \sigma\gamma\Theta x}_{\text{first-order terms}} + \sigma \underbrace{\sum_{s=2}^{\infty}(-1)^s \frac{\gamma^s\Theta^s}{s!}x^s}_{\text{higher-order terms}}, \tag{21}$$

which along with the Phillips curve (18b) determines global dynamics. As previously mentioned, close to the targeted steady state $(0,0)$, dynamics are dominated by the first-order terms. When $\phi_\pi > 1$, the first-order term involving inflation induces destabilizing dynamics, but the first-order term involving x induces stabilizing dynamics. For example, because the coefficient on x is negative ($-\gamma\Theta < 0$), a positive x causes $\dot{x}$ to become negative, i.e., it causes x to return to 0, given all else. Thus, when risk is countercyclical, ϕ_π need to be larger so that the destabilizing effect overwhelms the stabilizing effect. The cyclical-risk augmented Taylor principle (19) states how large ϕ_π needs to be for this to be the case. Since this stabilizing force is not present in RANK or in HANK with acyclical risk, $\phi_\pi > 1$ suffices.

The Taylor principle (19) ensures that locally around $(0,0)$, the net effect of the first-order terms is destabilizing and that the target equilibrium is locally determinate. When $\sigma = 0$ or $\Theta = 0$, the Taylor principle is also sufficient to ensure global determinacy. However, when $\Theta > 0$, while (19) ensures that local to $(0,0)$, the economy features explosive dynamics, but it does not guarantee this for (x,π) farther away from the targeted steady state. This is because when x is large in magnitude, the higher-order terms in (21) dominate the linear terms and are the primary drivers of dynamics. Proposition 3 shows that the higher order terms induce a stabilizing force which cannot be overwhelmed no matter how large ϕ_π is, and thus, there is global indeterminacy as long as $\Theta > 0$. The precise form these stabilizing dynamics take crucially depend on the magnitude of Θ, as Proposition 3 describes.

Mildly countercyclical risk When risk is mildly countercyclical $\Theta \in (0, \overline{\Theta})$, the Taylor principle (19) ensures that the targeted steady state $(0,0)$ is unstable. Consequently, any trajectory which starts in the neighborhood of $(0,0)$ initially diverges away from the targeted steady state. However, the presence of the untargeted steady state means that not all these trajectories explode. In fact, some of these *locally* explosive trajectories eventually get attracted by the untargeted steady state and converge to it. Thus, even a small shock which displaces the economy from the targeted steady state can cause it to transition to a permanently lower output and below-target inflation. Furthermore, as the black trajectory in Figure 3a shows, inflation and output can both be above target during the transition, but eventually converge to the steady state with lower economic activity and and below-target inflation.[29]

Moderately countercyclical risk For moderately countercyclical risk, i.e., $\Theta \in [\overline{\Theta}, \Theta^\star]$, the Taylor principle (19) still ensures that any trajectory starting in the neighborhood of the targeted steady state initially diverges away from it (gray-trajectory in Figure 3c), but as the economy gets farther away from $(0,0)$, the higher order terms tend to "*push-back*" towards $(0,0)$, causing these trajectories to converge to a stable limit-cycle, where the destabilizing effect of first-order terms are exactly offset by the stabilizing effect of higher-order terms. Thus, instead of converging to the untargeted steady state, any trajectory which starts in the neighborhood of the $(0,0)$ now converges to a stable-limit cycle (depicted by the black trajectory in Figure 3c).[30] This shows that even a small shock which dislodges the economy from the targeted steady state can cause the economy can get stuck in a cycle in which the output-gap and inflation are permanently away from their targeted values. Finally, the fact that the higher-order terms push the dynamics inwards towards $(0,0)$ is also visible from the orange trajectory, which starts near the untargeted steady state, but converges to the stable cycle.

This pattern of of dynamics described above is the same for any $\Theta \in (\overline{\Theta}, \Theta^\star)$, but the exact value of Θ affects the periodicity and amplitude of cycles. In particular, the periodicity of the cycles is a decreasing function of Θ in this range: as we lower Θ towards $\overline{\Theta}$, the amplitude of the cycles become larger. Given our calibration, $\Theta^\star = \rho/\sigma\gamma = 31.08$, implying that for $\Theta \in (15.8, 31.08)$, any trajectories which start close to the targeted steady state eventually converge to a stable cycle, which implies global indeterminacy. Furthermore, this range of values for Θ is likely the empirically relevant case, since it contains the estimates of Bilbiie et al. (2023), which translated to our model, imply that $\Theta \in [21.98, 29.8]$.

At the lower end of this region $\Theta = \overline{\Theta}$ (the boundary between mild and moderately countercyclical risk), the cycles described above get absorbed into a homoclinic orbit which has an infinite period, and is depicted in Figure 3b.[31] The upper end of this region, $\Theta = \Theta^\star$ (the boundary between moderately and highly countercyclical risk), a limit cycle exists but is degenerate: the limit cycle collapses onto the steady state $(0,0)$.[32] In both these cases, the equilibrium is globally indeterminate.

Highly countercyclical risk When risk is highly countercyclical, $\Theta > \Theta^\star$, Proposition 2 shows that even the Taylor principle (19) cannot render the targeted equilibrium locally determinate. As such, trajectories which start in the neighborhood of $(0,0)$ converge to it, implying that there are multiple bounded trajectories which satisfy equilibrium conditions (depicted by the grey trajectory in Figure 3d). Proposition 3 also guarantees the existence of a *saddle-connection* (depicted by the orange trajectory in Figure 3d), along which the economy can start close to the low economic activity steady state and

[29]While it is not possible to analytically characterize $\overline{\Theta}$, numerically we find that, given the rest of our calibration, $\overline{\Theta} \approx 15.8$, which suggest that this range is below the empirically relevant estimates range of Θ based on Bilbiie et al. (2023).

[30]The trajectories depicted are computed by setting $\Theta = 28.1$, which is Bilbiie et al. (2023)'s modal estimate of Θ.

[31]The homoclinic orbit is a path through phase space which joins the untargeted state $(\underline{x}, \underline{\pi}^g)$ to itself.

[32]By construction, at $\Theta = \Theta^*$, the Jacobian of the $(0,0)$ steady state undergoes a Hopf bifurcation: its eigenvalues have zero real parts. Therefore the local stability of that steady state depends on higher-order nonlinear terms. Since, following point 1 of Proposition 3, the cycles around $(0,0)$ are stable prior to collapsing steady state at the bifurcation point $\Theta = \Theta^*$, the targeted steady state $(0,0)$ is stable and indeterminate. The emergence and characterization of periodic cycles through a Hopf bifurcation around a steady state can be studied locally and independently of whether a second steady state or homoclinic orbits exist. For a global characterization of periodic orbits resulting in Hopf bifurcations, see Alexander and Yorke (1978).

converge to the targeted steady state, implying that even trajectories which start far away from the target steady state remain bounded and constitute valid equilbria.

Overall, the description above underscores the fact that the Taylor principle is not sufficient for ensuring global determinacy if risk is even mildly countercyclical. It is worth noting that our conclusion about the inability of the Taylor principle to deliver an unique equilibrium is different from the objections that John Cochrane has raised regarding the Taylor principle. Cochrane criticizes the Taylor principle on the basis that it guarantees a unique equilibrium by assuming away *unbounded* equilibria, and he argues that there is no economic reason to rule these out. In contrast, our characterization above shows that as long as risk is countercyclical, imposing the Taylor principle does not rule out the existence of multiple bounded equilibria even *if* we a priori rule out unbounded equilibria.

The diverging conclusions based on local vs global determinacy above have important implications regarding the design of monetary policy, which are particularly relevant in the current context. High inflation in the post COVID period has led many central banks to respond very strongly to the increase in inflation in order to bring inflation back down, as well as to keep inflation expectations anchored. If we were to design monetary policy based on the conclusions of the local determinacy analysis, one would conclude that a high enough ϕ_π, i.e., a large enough increase in the nominal rate in response to the higher inflation should be sufficient to ensure that the target equilibrium is determinate, or in other words, that inflation expectations remain anchored around the targeted level of inflation $\pi^\star$. However, Proposition 3 shows that this prescription is not sufficient to ensure that inflation expectations remain anchored. In particular, Proposition 3 shows that no matter how aggressively the central bank raises rates to counter higher inflation, i.e., no matter how large ϕ_π is, the economy can still get trapped in a situation where inflation expectations (as well as inflation) never return to their targeted level. Next, we discuss how other standard monetary policy rules also fail at implementing an unique equilbrium.

4 Do other standard policy rules do better?

The large literature studying local determinacy in RANK economies has shown that allowing policy rates to respond to output-gap fluctuations, or adding inertial behavior makes local determinacy more likely in RANK economies. Next, we show that while these changes to the purely-inflation targeting rule do make local determinacy more likely in our HANK economy as well, they still fail at delivering global determinacy.

4.1 Output-gap response

We first consider the case in which monetary policy also responds to output-gap fluctuations. The augmented rule can be written as:

$$i_t = \bar{r} + \pi^\star + \phi_\pi \left(\pi_t - \pi^\star\right) + \phi_x x_t, \tag{22}$$

which nests the inflation targeting rule (6) if we set $\phi_x = 0$. Given the interest rate rule (22), dynamics of the output-gap x_t, and inflation-gap π_t^g are described by:

$$\dot{x}_t = (\phi_\pi - 1)\,\pi_t^g + \phi_x x_t - \left(r^\star(x_t) - \bar{r}\right) \tag{23a}$$

$$\dot{\pi}_t^g = \rho\pi_t^g - \kappa\,(e^{x_t} - 1) \tag{23b}$$

How does ϕ_x affect local determinacy? As is well known in the RANK literature, allowing the nominal rate to respond to output-gap fluctuations in addition to inflation reduces the burden on ϕ_π to ensure that the targeted equilibrium is locally determinate (see, e.g. Bullard and Mitra (2002)). The same is true in the context of our HANK economy with countercyclical risk. Recall that in the inflation targeting rule (6) (with $\phi_x = 0$), a higher ϕ_π could only guarantee local determinacy only if risk was mildly or moderately countercyclical ($0 < \Theta < \Theta^\star$). However, if risk is highly countercyclical ($\Theta \geq \Theta^\star$), no finite ϕ_π can deliver local determinacy. However, as Proposition 4 shows, a monetary policy rule which responds to output-gap fluctuations can *always* guarantee local determinacy, no matter how countercyclical risk is.

Proposition 4 (Output-gap stabilization)**.** *For any $\Theta \geq 0$, the combination of a large enough ϕ_x and ϕ_π is sufficient for local determinacy of the targeted steady state. In particular, local determinacy requires:*

$$\phi_\pi > \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} \qquad \textit{and} \qquad \phi_x > \sigma\gamma\,(\Theta - \Theta^\star) \tag{24}$$

Proof. See Appendix C.1. □

Consistent with Proposition 2, (24) shows that if risk is for mild or moderately countercyclical, $\Theta \in (0, \Theta^\star]$, then $\phi_x = 0$ is sufficient for local determinacy as long as ϕ_π is large enough: $\phi_\pi > \varphi(\Theta)$. However, if risk is highly countercyclical ($\Theta > \Theta^\star$), setting ϕ_x slightly above 0 is not sufficient, and local determinacy requires $\phi_x > \sigma\gamma(\Theta - \Theta^\star) > 0$, alongside a large enough ϕ_π. Overall, (24) shows that the more countercyclical risk, the higher ϕ_π and ϕ_x need to be to ensure local determinacy; raising only one of the two is not sufficient. To understand why a high enough ϕ_x can guarantee local determinacy, it is again useful to concentrate on the IS curve. With $\phi_x \neq 0$, the IS curve (23a) can be written as:

$$\dot{x}_t = \underbrace{(\phi_\pi - 1)\,\pi_t^g + \phi_x x_t}_{\text{destabilizing}} - \sigma\gamma\Theta x_t + \mathcal{O}(x^2) \qquad \textit{for} \qquad (x, \pi^g) \to (0, 0)$$

Relative to the case with $\phi_x = 0$, the expression above shows that in addition to the $(\phi_\pi - 1)\pi$ term which supplies the destabilizing forces, now an additional term, $\phi_x x$, also generates destabilizing dynamics. When x is away from its steady state value, say $x > 0$, this term makes $\dot{x}$ more positive, thus pushing x further away from 0. The stabilizing force, provided by the term, $-\sigma\gamma\Theta x$ is the same as in the case with $\phi_x = 0$. Thus, as (24) shows, a large enough ϕ_π and ϕ_x make it easier for the destabilizing forces to overwhelm the stabilizing force, generating local determinacy.

While the combination of a high enough ϕ_π and ϕ_x helps resolve the problem of *local* determinacy, it does little to resolve the problem of global indeterminacy. This inability to eliminate global indeterminacy stems from the fact that even with $\phi_x > 0$, no matter how large, the untargeted steady state

$(\underline{x}, \underline{\pi}^g)$ continues to exist,[33] and features lower output than the targeted steady state as well as below target inflation. In fact, the larger the $\phi_x > 0$, the lower are output and inflation in the untargeted steady state. Figure 4 depicts this graphically, where the dotted-blue and the dotted-red curves denote the IS curve ($\dot{x} = 0$ nullcline) with $\phi_x > 0$ and the Phillips curve ($\dot{\pi}^g = 0$ nullcline), while the dotted-grey curve denotes the IS curve with $\phi_x = 0$.[34] A higher ϕ_x shifts the IS curve upwards relative to the baseline case (compare the blue-dotted curve to the grey-dotted curve in Figure 4). Even with $\phi_x > 0$, the IS curve intersects the $\dot{\pi}^g = 0$ nullcline twice, indicating the existence of two steady states.

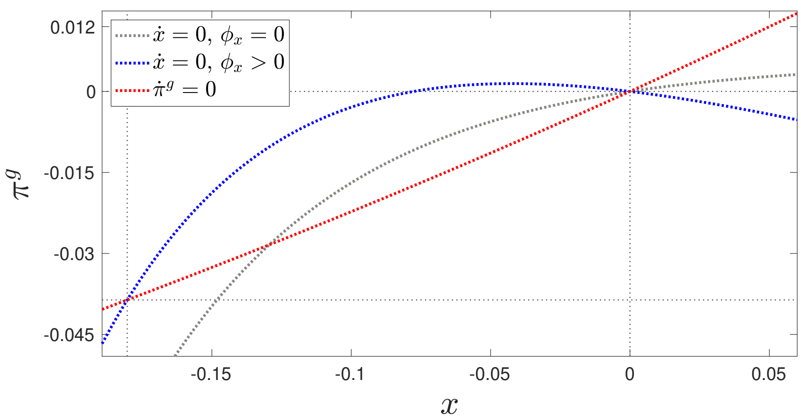


Figure 4: Nullclines with $\phi_x > 0$: The red-dotted curve denotes the $\dot{\pi} = 0$ nullcline, and is unchanged relative to the baseline model. The dotted-blue curve depicts the IS curve ($\dot{x} = 0$ nullcline) in the case with $\phi_x > 0$, while the dotted-grey curve depicts the IS curve in the baseline model with $\phi_x = 0$.

The existence of the second steady state again implies that the equilibrium is *globally* indeterminate, because the two steady states imply that there are at least two bounded trajectories which satisfy all equilibrium conditions: $(x_t, \pi_t^g) = (0, 0)$ and $(x_t, \pi_t^g) = (\underline{x}, \underline{\pi}^g)$. Moreover, as with $\phi_x = 0$, the untargeted steady state is not the only other bounded trajectory. Proposition 5 provides an exhaustive characterization of global dynamics for a given cyclicality Θ as a function of ϕ_x.

Proposition 5. *For a given $\Theta > 0$, assume that $\phi_\pi > \varphi(\Theta)$. Then, defining $\phi_x^\star = \sigma\gamma(\Theta - \Theta^\star)$, the global dynamics of the economy with monetary policy rule (22) can be split into 3 regions:*

1. $\phi_x < \phi_x^\star$ ***(small ϕ_x):*** *If $\phi_x < \phi_x^\star$, trajectories which start in the neighborhood of $(0, 0)$ converge to the the targeted steady state, and thus, the targeted equilibrium is both locally and globally indeterminate. In addition to the multiple trajectories originating near the targeted steady state which remain bounded, there also exists a saddle-connection along which the economy can transition from the neighborhood of the untargeted steady state to the targeted steady state. All trajectories which originate at any point on this saddle connection also remain bounded.*

2. $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$ ***(medium ϕ_x):*** *$\exists \overline{\phi}_x > \phi_x^\star$, such that for any $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$, the targeted equilibrium is locally determinate, but globally indeterminate. Trajectories which start in the neighborhood of the targeted steady state initially diverge away from it, but eventually remain bounded and converge to a stable limit-cycle surrounding the targeted steady state. The amplitude and periodicity of the cycles is increasing in ϕ_x*

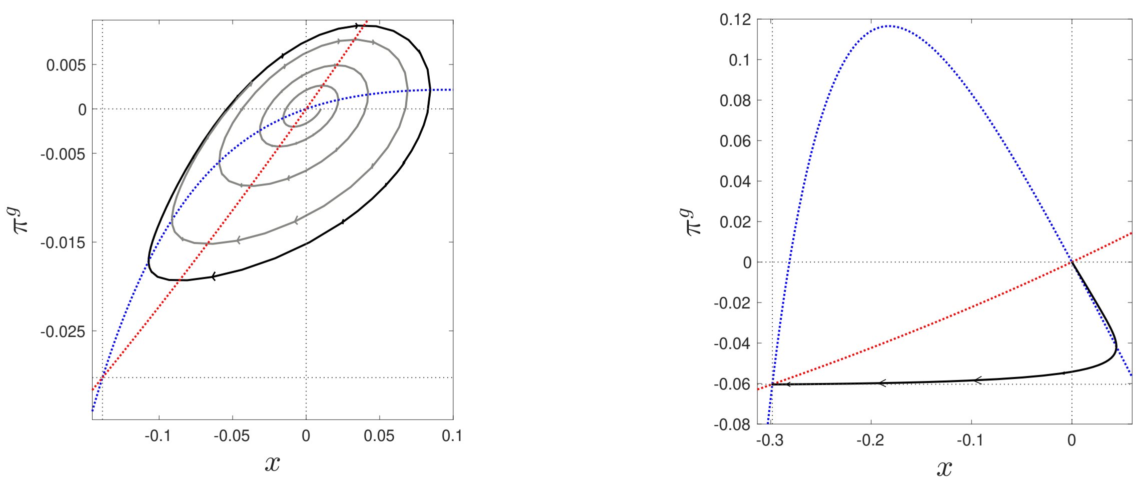
[33] See Appendix C.2 for a proof.
[34] The value of ϕ_x does not shift the $\dot{\pi}^g = 0$ nullcline.

in this range. At the upper limit of this interval, $\phi_x = \overline{\phi}_x$, the limit-cycles are absorbed into a homoclinic orbit. At the other end of this range, $\phi_x = \phi_x^\star$, the limit-cycles collapse onto the targeted steady state. The equilibrium is still globally indeterminate since the higher-order terms ensure that any trajectory starting in the neighborhood of $(0,0)$ converge back to $(0,0)$.

3. $\phi_x > \overline{\phi}_x$ ***(large*** ϕ_x***):*** *For $\phi_x > \overline{\phi}_x$, the targeted equilibrium is locally determinate but still globally indeterminate. There are no stable limit-cycles in this range of ϕ_x, but the equilibrium is still globally indeterminate, owing to the existence of a saddle-connection, along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. Any trajectory which originates on this saddle connection also remains bounded.*

No matter how large ϕ_π and ϕ_x are, the equilibrium is still globally indeterminate if risk is even mildy countercyclical.

Proof. See Appendix C.3. □



(a) medium ϕ_x (b) large ϕ_x

Figure 5: Global dynamics with $\phi_x > 0$

Proposition 5 shows that no combination of π_π and ϕ_x can ensure global determinacy. Figure 5 depicts the non-fundamental fluctuations that can emerge in or HANK economy. To plot this Figure, we set $\Theta = 28.1$, which is the model estimate of Bilbiie et al. (2023).[35] As has been the convention, in Figure 5, the dotted-red curve depicts the $\dot{\pi} = 0$-nullcline and the dotted-blue curve depicts the $\dot{x} = 0$-nullcline. Figure 5a plots global dynamics with $\phi_x = 0.01$, which lies in the range $\phi_x \in (\phi_x^\star, \overline{\phi}_x)$: in this range, trajectories starting in the neighborhood of $(0,0)$ (dark gray trajectory) converges to the stable cycle (black trajectory).[36] In contrast, Figure 5b features a larger $\phi_x = 0.5$, which is the standard calibration of the Taylor rule (Taylor, 1993) and shows that there exists a saddle connection from the targeted to the untargeted steady state. A trajectory beginning at any point on this saddle-connection

[35] Since $\Theta = 28.1 < 31.1 = \Theta^\star$, we have $\phi_x^\star = \sigma\gamma(\Theta - \Theta^\star) = -0.0041$. In other words, the *small* ϕ_x region corresponds to the interval $\phi_x \in (-\infty, -0.0041)$, which we ignore, because the sensible range of ϕ_x is in the interval $[0, \infty)$. Consequently, Figure 5 only plots dynamics in the *medium* ϕ_x and *large* ϕ_x cases.

[36] In addition, trajectories originating near the untargeted steady state also converge to the cycle as in Figure 3c, but we omit these trajectories from Figure 5a to avoid clutter.

remains bounded, and constitutes a bounded sequence which satisfies equilibrium. The same is true for more countercyclical risk, $\Theta > \Theta^{\star}$.[37] Overall, while allowing for a large enough ϕ_x can guarantee local determinacy, the equilibrium is globally indeterminate in our HANK economy with countercyclical risk, no matter how large ϕ_π and ϕ_x are.

4.2 Inertial rules

The monetary policy rule we have studied so far only react to changes in current economic conditions. However, empirically, many central banks have been noted to have a tendency to only adjust the policy rate gradually in response to changes in economic conditions. Such inertial rules have been shown to be desirable from the point of view of delivering local determinacy (Bullard and Mitra, 2007), and for various other reasons (Woodford, 2003b). In continuous time, such a rule can be written as:[38]

$$di_t = \alpha \Big[i_t - \bar{r} - \phi_\pi(\pi_t - \pi^{\star}) \Big] dt, \tag{25}$$

where α controls the relative weight on inflation in the past relative to current inflation. A smaller α in (26) implies a larger weight on past inflation relative to current inflation, and that the rule is *more backward-looking*. In fact, the limit $\alpha \to 0$ corresponds to the *price-level targeting* (PLT) limit. In the limit $\alpha \to \infty$ converges to the policy rule (6), which only responded to changes in current inflation, and is not backward-looking at all. Instead of working with (25), we transform it into the equivalent *average-inflation targeting* (AIT) rule:

$$i_t = \bar{r} + \phi_\pi(\pi^b_t - \pi^{\star}), \qquad \text{where} \qquad \pi^b_t = \alpha \int_{-\infty}^{t} e^{-\alpha(t-\tau)} \pi_\tau d\tau \qquad \text{for} \qquad \alpha \in (0, \infty), \tag{26}$$

(26) shows that the inertial behavior is equivalent to the central bank responding to changes in the weighted-average of past and current inflation (denoted π^b_t), instead of just changes in current inflation. The dynamics of the economy under an AIT rule are described by a 3-dimensional system of ODEs:

$$\dot{x}_t = \phi_\pi(\pi^b_t - \pi^{\star}) - (\pi_t - \pi^{\star}) - (r^{\star}(x_t) - \bar{r}) \tag{27}$$

$$\dot{\pi}_t = \rho(\pi_t - \pi^{\star}) - \kappa\left(e^{x_t} - 1\right) \tag{28}$$

$$\dot{\pi}^b_t = \alpha\left(\pi_t - \pi^b_t\right), \tag{29}$$

where (29) is derived by taking a time-derivative of π^b_t described in (26). Imposing steady state in (27)-(29), it is easy to see that $x = 0$ and $\pi = \pi^b = \pi^{\star}$ is still a steady state (the targeted steady state).

Compared to the purely inflation targeting rule (6), a sufficiently backward-looking inertial rule (26) can guarantee local-determinacy of the targeted steady state, even if risk is highly countercyclical

[37] While Figure 3 showed global dynamics when $\Theta < \Theta^{\star}$, Figure 7 in Appendix C.4 plots dynamics for $\Theta > \Theta^{\star}$, and graphically depicts the global dynamics of the output-gap and inflation-gap as a function of ϕ_x.

[38] This rule is analogous to the more familiar discrete-time specification for such rules:

$$1 + i_t = \varrho\left(1 + i_{t-1}\right) + (1 - \varrho)\left[1 + \bar{r} + \phi_\pi(\pi_t - \pi^{\star})\right],$$

where $\varrho \in (0, 1)$ captures the idea that the policy rate at date t displays inertia: $1 + i_t$ depends not just on the deviation of current inflation from target, but also depends on what the policy rate was set to in the past $1 + i_{t-1}$.

$(\Theta > \Theta^\star)$. This is formalized in Proposition 6.[39]

Proposition 6 (Local determinacy with an inertial rule)**.** *For a given $\Theta > 0$, and any π_0^b in the small neighborhood of the targeted steady state $x = 0$, $\pi^g = \pi^b = \pi^\star$, there exists a unique $\{x_0, \pi_0^g\}$, such that the trajectory $\{x_t, \pi_t, \pi_t^b\}_{t=0}^\infty$ remains bounded inside this neighborhood, as long as α is sufficiently close to 0 and $\phi_\pi > \varphi(\Theta)$. In other words, the targeted equilibrium is locally determinate if*

$$\phi_\pi > \varphi(\Theta) \qquad \textit{and} \qquad \alpha \in \Big[0, \alpha^\star(\theta)\Big),$$

If risk is mildly or moderately countercyclical $\alpha^\star(\Theta) = \infty$, but if risk is highly countercyclical, then $\alpha^\star(\Theta) < \infty$ (where the exact expression is available in Appendix D.1).

Proof. See Appendix D.1. □

Proposition 6 shows that when risk is mildly or moderately countercyclical $(\Theta < \Theta^\star)$, any $\alpha \geq 0$ delivers local determinacy, as long as $\phi_\pi > \varphi(\Theta)$. This follows directly from the fact that in the limit as $\alpha \to \infty$, the inertial rule (26) converges to the policy rule (6), and we know from Proposition 2 that if $0 < \Theta < \Theta^\star$, then $\phi_\pi > \varphi(\Theta)$ is sufficient for local determinacy. However, even when $\Theta \geq \Theta^\star$, a small enough α ensures that the targeted equilibrium is locally determinate. In fact, a corollary of this result is that in the PLT limit, $\alpha \to 0$, the targeted equilibrium is always locally determinate regardless of how countercyclical risk may be.[40] Thus, a sufficiently backward-looking inertial rule ensures local determinacy of the target equilibrium, no matter how countercyclical risk is.

However, this improved performance in terms of ensuring local determinacy does not translate into global determinacy. In fact, Proposition 7 shows that as long as risk is countercyclical, the equilibrium is always globally indeterminate, no matter how backward-looking the rule.

Proposition 7. *Suppose that risk is countercyclical $(\Theta > 0)$ and monetary policy is described by the inertial rule (26) satisfying $\phi_\pi > \varphi(\Theta)$. Then the equilibrium is always globally indeterminate, regardless of the magnitude of $\alpha \in [0, \infty)$. The global dynamics under the inertial rule (26) can be divided into three regions based on the magnitude of $\alpha \geq 0$:*

1. ***Mildly backward-looking** $\Big(\alpha > \alpha^\star(\Theta)\Big)$**:** For large α, the targeted equilibrium is both locally and globally indeterminate. Not only do trajectories which originate in the neighborhood of the targeted steady state converge to it, there also exists a saddle connection along which trajectories which originate near the untargeted steady state converge to the targeted steady state.*

[39] With an inertial rule, the definition of local determinacy is slightly different relative to our baseline model. This is because our baseline model featured two forward looking variables x, π^g and so local determinacy required two eigenvalues with positive real parts. In contrast, the inertial rule adds as *predetermined* variable π^b, and so local determinacy requires two eigenvalues with positive real parts, and one negative eigenvalue. In other words, local determinacy of the targeted equilibrium now requires that for a given value of the predetermined variable π_0^b in a small neighborhood of the targeted steady state, there exist a unique (x_0, π_0^g) starting from which the trajectory $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ satisfies all equilibrium conditions *and* remains bounded. Global determinacy, then, requires that from *any* given value of the predetermined variable π_0^b (not necessarily in the neighborhood of the targeted steady state), there exists a unique (x_0, π_0^g), starting from which the trajectory $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ remains bounded while satisfying all equilibrium conditions.

[40] Bilbiie (forthcoming) also shows that a PLT rule guarantees local-determinacy in this THANK model. However, that paper does not study whether such a rule ensures global determinacy.

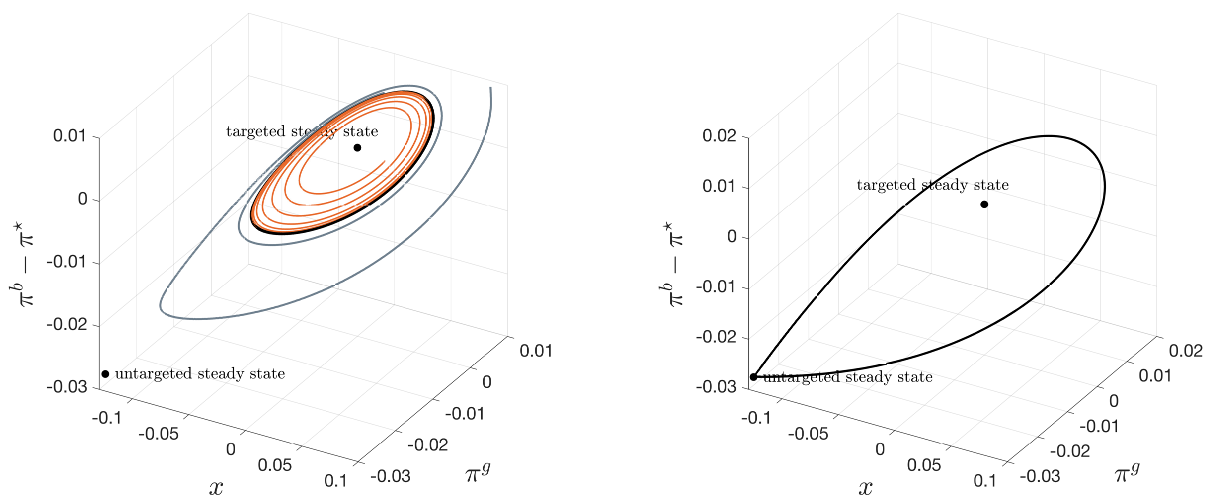
2. ***Moderately backward-looking*** $\left(\alpha \in [\underline{\alpha}(\Theta), \alpha^\star(\Theta)]\right)$: *$\exists \underline{\alpha}(\Theta) < \alpha^\star(\Theta)$ such that for any $\alpha \in (\underline{\alpha}(\Theta), \alpha^\star(\Theta))$, any trajectory which originates near the targeted steady state initially diverges but then converges to a stable cycle which surrounds the targeted steady state. The amplitude of these cycles is a decreasing function of α in this region. Overall, for $\alpha \in (\underline{\alpha}(\Theta), \alpha^\star(\Theta))$, the targeted equilibrium is locally determinate but there is global indeterminacy. At the lower boundary $\alpha = \underline{\alpha}(\Theta)$, the stable cycles are absorbed into a homoclinic orbit, and at the upper boundary $\alpha = \alpha^\star(\Theta)$, the limit cycles are degenerate, but there is still global indeterminacy since the higher-order terms push any trajectory starting near the targeted steady state back towards it.*

3. ***Strongly backward-looking*** $\left(\alpha < \underline{\alpha}(\Theta)\right)$: *For small enough α, there are no stable cycles. However, there exists a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. Thus, a small α ensures local determinacy but not global determinacy.*

Overall, if risk is countercyclical, there is global indeterminacy, no matter how backward-looking the policy rule.

Proof. See Appendix D.4. □

Proposition 7 characterizes the global dynamics of the economy under the average inflation targeting rule for all degrees of backward-lookingness, and shows that no matter how backward looking a policy rule, it cannot eliminate global indeterminacy. Figure 6 plots global dynamics for $\Theta < \Theta^\star$.



(a) moderately backward-looking

(b) homoclinic orbit ($\alpha = \underline{\alpha}$)

Figure 6: Global dynamics with an inertial rule

When $\Theta = 28.1 < \Theta^\star$, $\alpha^\star(\Theta) = \infty$ and set $\alpha \geq \alpha^\star(\Theta)$ is empty. Thus, the global dynamics in this case are described by points 2 and 3 of Proposition 7. Figure 6a depicts the dynamics for a moderately backward-looking rule, where we have set $\alpha = 9$, which is larger than $\underline{\alpha}(\Theta) = 1.03$ and smaller than $\alpha^\star(\Theta) = \infty$. The figure shows that trajectories which originate near the targeted steady state (orange curves) or even further away from the steady state (grey trajectory), both converge to a stable cycle (black trajectory). Figure 6b depicts the homoclinic orbit which occurs if $\alpha = \underline{\alpha}(\Theta) = 1.03$.[41]

Trajectories which originate inside the homoclinic orbit converge to it and remain bounded. While point 3 of Proposition 7 guarantees that for $\alpha \in [0, \underline{\alpha})$, there exists a saddle connection (along which the economy can move from the neighborhood of the targeted steady state to the untargeted steady state), it is hard to numerically plot this trajectory because it is hard to numerically compute the 1 dimensional stable manifold. Hence we are unable to plot the dynamics described in point 3. Overall, the discussion above shows that as long as risk is countercyclical ($\Theta > 0$), there is global indeterminacy, no matter how backward looking the policy rule is.

5 Designing effective monetary policy rules

Our analysis has shown that standard monetary policy rules cannot guarantee global determinacy in our HANK economy with countercyclical risk, no matter how large ϕ_π is, whether or not they put a large enough weight on output-gap stabilization or if they are sufficiently backward looking. The key force generating indeterminacy is that in our HANK economy with countercyclical risk, the natural rate $r^\star(y)$ is endogenous and co-moves with output, and this opens the economy up to the possibility of self-fulfilling fluctuations. Standard monetary policy rules fail to deliver global determinacy because they do not prevent such self-fulfilling beliefs from taking root. As we show next, this is because the intercept in all the policy rules studied above was the flexible-price real interest rate $\bar{r}$.

As is well known from the RANK literature, monetary policy should track the flexible-price real interest rate (neutral rate) to eliminate the effects of exogenous demand shocks. Since our model abstracts from aggregate risk $\bar{r}_t = \bar{r}$ is constant. More generally, it would change over time in response to exogenous shocks, e.g., a shock to the discount rate ρ_t. But importantly, even when it is time-varying, it does *not* depend endogenously on the level of output. In order to optimally offset the effects of these *exogenous* fluctuations in $\bar{r}_t$, a Taylor type policy rule must feature a time-varying intercept which is equal to $\bar{r}_t$ (Galí, 2015):

$$i_t = \bar{r}_t + \pi^\star + \phi_\pi(\pi_t - 1), \tag{30}$$

where the t-subscript on $\bar{r}$ is meant to denote that the neutral rate need not be constant. By setting the intercept of (30) to the flexible-price real interest rate, monetary policy tracks changes in the neutral rate one-for-one and completely undoes the effects of exogenous demand shocks in RANK, but this has no consequence for the determinacy of equilibrium in RANK.

However, in our HANK economy with countercyclical risk, the economy is susceptible to "*endogenous* demand shocks". For example, suppose that ξ_h households believe that the economy is going to enter a recession, and consequently they face a higher probability of transitioning to the ξ_l state. Holding all else constant, this increases households' desired precautionary savings demand, pushing down the natural rate. If monetary policy does not respond sufficiently to this downward movement

[41] The existence of homoclinic orbits in 3 dimensional systems can also give rise chaotic dynamics, as studied by Shilnikov (See Chapter 6 of Kuznetsov (1998)). Barnett et al. (2022) study an application of Shilnikov chaos to a New Keynesian model.

in $r^\star(x)$, the real interest rate is higher than the natural rate $r > r^\star$, incentivizing households to reduce spending. This can be seen via IS curve (14), which show that when $r_t > r_t^\star$, output (gap) growth is positive: $\dot{x} = \dot{y}/y > 0$. In other words, given that $r_t > r_t^\star$, households reduce their current consumption, and because of nominal rigidities, this results in lower output, rendering the initial belief self-fulfilling. This acts as a negative endogenous demand shock.

Now suppose that instead of (30), the monetary policy rule is described by

$$i_t = r^\star(x_t) + \pi^\star + \phi_\pi(\pi_t - 1), \tag{31}$$

The key change in a policy rule of the form (31) relative to (30), is that in addition to adjusting the nominal rate in response to changes in inflation, monetary policy also responds to *endogenous* changes in the natural rate $r^\star(x)$ one-for-one. With policy rule (31), monetary policy tracks the lower natural rate $r^\star(x)$ one-for-one, by lowering i_t by the same amount as the decline in $r^\star(x)$. This lower interest rate undoes the desire to increase precautionary savings, leaving current spending unchanged. Since households do not reduce current consumption, lower output cannot be supported in equilibrium. Hence the initial beliefs about the economy entering a recession cannot be self-fulfilling, and the economy remains at the targeted steady state. Thus, by responding to endogenous fluctuations in $r^\star(y)$, monetary policy eliminates the endogenous demand shock.

It is worth noting that even when the neutral rate is time-varying in RANK, sometimes the monetary rule in RANK is sub-optimally specified with a constant intercept, which is equal to the flexible price real interest rate in steady state. While doing so results in sub-optimal dynamics and welfare losses in response to demand shocks, it does not affect the determinacy properties of the RANK economy. However, this is different from our HANK economy, where setting the "incorrect" intercept can result in global indeterminacy alongside the welfare losses from the non-fundamental fluctuations caused by the endogenous demand shocks. Overall, just like an appropriate monetary policy response can undo the effects of an exogenous demand shock in RANK, it can also eliminate endogenous demand shocks from generating non-fundamental fluctuations in a HANK economy with countercyclical risk. This idea is formalized in Proposition 8 below.

Proposition 8. *Suppose that monetary policy is described by* (31)*. Then, for any* $\Theta > 0$*, the targeted equilibrium is globally determinate as long as* $\phi_\pi > 1$*.*

Proof. See Appendix E. □

Proposition 8 shows that once the problem is diagnosed correctly, the simple Taylor principle $\phi_\pi > 1$ is sufficient for both local and global determinacy regardless of how countercyclical risk is.[42] To see why this is true, notice that with the policy rule (31), the dynamics of the output-gap and inflation-gap are given by

$$\dot{x}_t = (\phi_\pi - 1)\,\pi_t^g \tag{32a}$$
$$\dot{\pi}_t^g = \rho\pi_t^g - \kappa\,(e^{x_t} - 1)\,, \tag{32b}$$

[42] Of course, in making this statement, we are still abstracting from the presence of a ELB. As in RANK, introducing an ELB would lead to global indeterminacy in the HANK economy as well.

which are the same as in the RANK benchmark. Consequently, since the simple Taylor principle ($\phi_\pi > 1$) is sufficient for equilibrium uniqueness in RANK, the same is true here.

The difference in the intercepts of the two policy rules (30) and (31) are key to understanding the different performances of the two rules in ensuring global determinacy when risk is countercyclical. In RANK, (30) and (31) have the same intercept because $r^\star(x)$ is independent of x and is always equal to $\bar{r}$, implying that both rules deliver a unique equilibrium as long as $\phi_\pi > 1$. However, in HANK with $\Theta > 0$, $r^\star(x)$ endogenously depends on the level of economic activity and is only equal to $\bar{r}$ if $x = 0$. Policy rule (31) delivers determinacy because it responds to the endogenous fluctuations in the natural rate, while (30) does not since it only tracks exogenous fluctuations in the natural rate, a difference that is reflected in the intercepts of the two rules.

Even though the two rules have different intercepts in general, the two rules have the same intercept *on-equilibrium*, even when risk is countercyclical. This is because the rule (31) ensures that the only bounded equilibrium features $x_t = 0$ for all t, and hence the natural rate $r_t^\star = \bar{r}$. Thus, one can think of the rule (31) as one under which monetary policy commits to change the nominal rate one-for-one with the natural rate if there was ever any change in the natural rate. This *off-equilibrium* commitment to adjust monetary policy ensures that beliefs about higher or lower output cannot be self-fulfilling.

While the lessons in this section might seem abstract and hard to implement in practice, it does not constitute a large deviation from the way in which most central banks set monetary policy. Following the centrality of inflation expectations for determinacy in the RANK framework, most central banks vigilantly monitor measures of long-run inflation expectations and stand ready to tighten policy if inflation expectations start to drift above the inflation target. In the same way, our HANK economy with countercyclical risk provides a rationale for central banks to pay equal attention to private sector beliefs about real activity. Our framework suggests that, just as central banks monitor and react to inflation expecations, if measures of confidence in the real economy (such as consumer confidence, households' perceived probability of job loss etc.) begin to drift down or up, monetary policy should act aggressively to reverse such beliefs. Simply trying to keep inflation expectations on target, while ignoring expectations about real activity, can enable self-fulfilling beliefs which lead to non-fundamental fluctuations in output and inflation.

6 Conclusion

In this paper, we show that if risk is even mildly countercyclical, HANK economies typically feature multiple equilibria under standard monetary policy rules: the Taylor principle, even if it is augmented to account for cyclical risk, is not sufficient to rule out the existence of multiple bounded equilibria. Similarly, standard policy rules which respond to output-gap fluctuations (in addition to inflation) also cannot rule out multiple equilibria, and neither can policy rules which incorporate inertial behavior. This is because in HANK economies with countercyclical risk, the natural interest rate is endogenous and co-moves with output. Policy needs to commit to adjusting nominal rates one-for-one with any endogenous fluctuations in the natural rate. Doing so implements a unique equilibrium by ensuring that non-fundamental beliefs about higher or lower output cannot be self-fulfilling.

Importantly, the source of this multiplicity of equilibria identified in this paper does not stem from the the presence of the ELB, and thus may plague the economy even during a tightening cycle. Furthermore, our analysis stresses that large and aggressive rate hikes in response to higher inflation *do not* ensure that inflation expectations will remain anchored around its target level. Our framework suggests that, just as central banks monitor and react to inflation expectations, if measures of confidence in the real economy (such as consumer confidence, households' perceived probability of job loss etc.) begin to drift down or up, monetary policy must act aggressively to reverse such beliefs, if it hopes to keep expectations anchored.

Finally, our analysis also shows that local stability analysis (which is often used to check the existence of multiple equilibria), can be misleading in HANK economies with countercyclical risk. In our economy, even when the targeted equilibrium is locally determinate, there is global indeterminacy and multiple bounded equilibria exist. This suggests that researchers using HANK models need to be more vigilant regarding the possibility of multiple equilibria.

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Appendix

A Derivation of IS curve

A.1 Household problem

Instead of directly solving the household problem in continuous time, we solve it in the discrete time limit in which each period is Δ units of time long. Then we derive the optimal decisions in continuous time by taking limits as $\Delta \to 0$. In discrete time, the problem of the household can be written as, where we have discarded the j subscript for convenience:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho \Delta t} \left[\frac{c_{t\Delta}^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_{t\Delta} \right] \Delta$$

s.t.

$$a_{t+\Delta} - a_t = (1 + r_t \Delta) \left(\xi_h w_t n_t + D_t - c_t \right) \Delta + r_{t+\Delta} \Delta a_t,$$

This problem can be formulated as a Bellman equation. The value function of a household with idiosyncratic productivity ξ_h and wealth a_t can be written as:

$$V(a_t, \xi_h) = \left[\frac{c_t^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_t \right] \Delta + (1-\rho\Delta) \left[(1 - \lambda_{l,t+\Delta}\Delta) V(a_{t+\Delta}, \xi_h) + \lambda_{l,t+\Delta} \Delta V(a_{t+\Delta}, \xi_l) \right], \tag{a.1}$$

where $a_{t+\Delta}$ is given by:

$$a_{t+\Delta} = (1 + r_t \Delta) \left[\left(\xi_h w_t n_t + D_t - c_t \right) \Delta + a_t \right] \qquad \text{and} \qquad a_{t+\Delta} \geq -\underline{a}, \tag{a.2}$$

and we have used the fact that for small Δ, $e^{-\rho\Delta} = 1 - \rho\Delta$. Similarly, the value function for a household with idiosyncratic productivity ξ_l and wealth a_t can be written as:

$$V(a_t, \xi_l) = \left[\frac{c_t^{1-\gamma^{-1}}}{1-\gamma^{-1}} - \psi n_t \right] \Delta + (1-\rho\Delta) \left[(1 - \lambda_{h,t+\Delta}\Delta) V(a_{t+\Delta}, \xi_l) + \lambda_{h,t+\Delta} \Delta V(a_{t+\Delta}, \xi_h) \right],$$

where $a_{t+\Delta}$ is given by:

$$a_{t+\Delta} = (1 + r_t \Delta) \left[\left(\xi_l w_t n_t + D_t - c_t \right) \Delta + a_t \right] \qquad \text{and} \qquad a_{t+\Delta} \geq -\underline{a},$$

The optimal choice of hours worked n_t by a household with idiosyncratic productivity ξ_h is given by:

$$\frac{\psi}{\xi_h w_t} = (1 + r_t \Delta)(1 - \rho\Delta) \left[(1 - \lambda_{l,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_h) + \lambda_{l,t+\Delta} \Delta V_a(a_{t+\Delta}, \xi_l) \right], \tag{a.3}$$

and by a household with idiosyncratic productivity ξ_l is given by:

$$\frac{\psi}{\xi_l w_t} = (1 + r_t\Delta)(1 - \rho\Delta)\left[(1 - \lambda_{h,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_l) + \lambda_{h,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_h)\right], \tag{a.4}$$

The optimal choice of consumption for a household with productivity ξ_h and ξ_h can be written as:

$$\begin{aligned} c_t(a, \xi_h) &\leq \left\{(1 + r_t\Delta)(1 - \rho\Delta)\left[(1 - \lambda_{l,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_h) + \lambda_{l,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_l)\right]\right\}^{-\gamma} && \text{(a.5)} \\ c_t(a, \xi_l) &\leq \left\{(1 + r_t\Delta)(1 - \rho\Delta)\left[(1 - \lambda_{h,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_l) + \lambda_{h,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_h)\right]\right\}^{-\gamma}, && \text{(a.6)} \end{aligned}$$

where we have divided both sides of each equation by Δ. The inequality captures the fact that households may be borrowing constrained. Next, the envelope conditions for ξ_h and ξ_l households are

$$\begin{aligned} V_a(a_t, \xi_h) &= (1 + r_t\Delta)(1 - \rho\Delta)\left[(1 - \lambda_{l,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_h) + \lambda_{l,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_l)\right] && \text{(a.7)} \\ V_a(a_t, \xi_l) &= (1 + r_t\Delta)(1 - \rho\Delta)\left[(1 - \lambda_{h,t+\Delta}\Delta) V_a(a_{t+\Delta}, \xi_l) + \lambda_{h,t+\Delta}\Delta V_a(a_{t+\Delta}, \xi_h)\right] && \text{(a.8)} \end{aligned}$$

Using the envelope conditions along with (a.3)–(a.6), we have:

$$c_{h,t} \equiv c_t(a, \xi_h) = \left(\frac{\xi_h w_t}{\psi}\right)^{\gamma} \quad \text{and} \quad c_{l,t} \equiv c_t(a, \xi_l) = \left(\frac{\xi_l w_t}{\psi}\right)^{\gamma}, \tag{a.9}$$

which shows that the consumption of all households with the same idiosyncratic productivity ξ_j is the same, regardless of their financial wealth.

Next, it is easy to see that in equilibrium, the ξ_l households are going to be borrowing constrained, and choose $a_{t+\Delta} = -\underline{a}$. In contrast, the ξ_h households are on their Euler equation and for asset markets to clear, they must save $\eta\underline{a}$ as a whole. To derive their Euler equation, we can rearrange the envelope condition for ξ_h households as:

$$\frac{\rho - r_t(1 - \rho\Delta)}{(1 + r_t\Delta)(1 - \rho\Delta)} V_a(a_t, \xi_h) = \left[\frac{V_a(a_{t+\Delta}, \xi_h) - V_a(a_t, \xi_h)}{\Delta} + \lambda_{l,t+\Delta}\left\{V_a(a_{t+\Delta}, \xi_l) - V_a(a_{t+\Delta}, \xi_h)\right\}\right]$$

Taking the limit of this equation as $\Delta \to 0$, we get:

$$(\rho - r_t) V_a(a_t, \xi_h) = \dot{V}_a(a_t, \xi_h) + \lambda_{l,t}\left\{V_a(a_t, \xi_l) - V_a(a_t, \xi_h)\right\} \tag{a.10}$$

Next, using (a.3)-(a.6), we have:

$$V_a(a_t, \xi_h) = c_{h,t}^{-\gamma^{-1}} \quad , V_a(a_t, \xi_l) = c_{l,t}^{-\gamma^{-1}} \quad \text{and} \quad \dot{V}_a(a_t, \xi_h) = c_{h,t}^{-\gamma^{-1}} \frac{\dot{c}_{h,t}}{c_{h,t}}$$

Using this in (a.10), we have the Euler equation:

$$\frac{\dot{c}_{h,t}}{c_{h,t}} = \gamma(r_t - \rho) + \gamma\lambda_{l,t}\left[\left(\frac{c_{l,t}}{c_{h,t}}\right)^{-\frac{1}{\gamma}} - 1\right],$$

which is the same as (9) in the main text. □

A.2 Constant fraction of low and high skilled households

At any date t, the fraction of households with skill ξ_l is given by the η_t, and evolves according to:

$$\dot{\eta}_t = -\lambda_{h,t}\eta_t + \lambda_{l,t}\Big(1-\eta_t\Big) \tag{a.11}$$

For a constant η, we need $\dot{\eta} = 0$ for all t, which requires:

$$\lambda_{h,t}\eta = \lambda_{l,t}\Big(1-\eta\Big) \tag{a.12}$$

Since $\lambda_{l,t}$ is time varying: $\lambda_{l,t} = \overline{\lambda}_l y_t^{-\Theta}$, for (a.12) to be satisfied, we need

$$\lambda_{h,t} = \overline{\lambda_h} y_t^{-\Theta}, \qquad \text{where} \qquad \overline{\lambda}_h = \overline{\lambda}_l\left(\frac{1-\eta}{\eta}\right) \tag{a.13}$$

Consequently, η is constant over time and is given by

$$\eta = \frac{\overline{\lambda}_l}{\overline{\lambda}_l + \overline{\lambda}_h}$$

□

A.3 Deriving a relationship between y and w

Goods market clearing at any date can be written as:

$$y_t \quad = \quad (1-\eta)c_{h,t} + \eta c_{l,t} \tag{a.14}$$

where η is a constant, as shown in Appendix A.2. Using the expressions for c_h and c_l in (a.9), (a.14) can be re-written as:

$$y_t \quad = \quad \left[(1-\eta)\left(\frac{\xi_h}{\psi}\right)^{\gamma} + \eta\left(\frac{\xi_l}{\psi}\right)^{\gamma}\right] w_t^{\gamma}$$

Normalizing $\psi = \left[(1-\eta)\xi_h^{\gamma} + \eta\xi_l^{\gamma}\right]^{\frac{1}{\gamma}}$, we can rewrite this as:

$$w_t \quad = \quad y_t^{\frac{1}{\gamma}}, \tag{a.15}$$

which is the same as (11) in the main text. Furthermore, setting $\dot{\pi} = 0$ and $\pi = \pi^{\star}$ in the Phillips curve (5) implies that real wages in the steady state with on-target inflation is $w = 1$. Consequently, (a.15) implies that output in the targeted steady state is $y = 1$. Finally, using this relationship in (a.9), we can express the per-capita consumption of households with skill ξ in terms of output at any date t:

$$c_{h,t} = \frac{\xi_h^{\gamma}}{(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} y_t > \frac{\xi_l^{\gamma}}{(1-\overline{\eta})\xi_h^{\gamma} + \overline{\eta}\xi_l^{\gamma}} y_t = c_{l,t} \tag{a.16}$$

□

A.4 The IS curve and the Phillips curve

Taking the time-derivative of expression for $c_{h,t}$ in (a.16) yields $\frac{\dot{c}_{h,t}}{c_{h,t}} = \frac{\dot{y}_t}{y_t}$. We can then rewrite (9) as:

$$\frac{\dot{y}_t}{y_t} = \gamma\Big(i_t - \pi_t - \rho\Big) + \gamma\overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right) y_t^{-\Theta}$$

Next, defining $x_t = \frac{1}{\gamma}\ln y_t$ and $\pi_t^g = \pi_t - \pi^\star$, we can rewrite the above as:

$$\dot{x}_t = i_t - \pi^\star - \pi_t^g - \rho + \overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right) e^{-\gamma\Theta x_t}$$

Substituting out i_t using the monetary policy rule (6), we have

$$\dot{x}_t = \overline{r} + (\phi_\pi - 1)\pi_t^g - \rho + \overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right) e^{-\gamma\Theta x_t}, \tag{a.17}$$

where $\overline{r}$ is the intercept in the monetary policy rule and also denotes the real interest rate in the targeted steady state. Similarly, rewriting the Phillips curve (5) in terms of π_t^g and x_t, we have

$$\dot{\pi}_t^g = \rho\pi_t^g - \kappa\left(e^{x_t} - 1\right)$$

Flexible-price limit Since we abstract from aggregate shocks, in the flexible-price limit of our economy ($\kappa \to \infty$), the Phillips curve implies that at any date t, $x_t = \pi_t^g = 0$. Using this in (a.17) and rearranging, the real interest rate in the flexible price limit $\overline{r}$ is given by $\overline{r} = \rho - \sigma$, where $\sigma = \overline{\lambda}_l\left(\frac{\xi_h}{\xi_l} - 1\right) \geq 0$ captures the effect of consumption risk faced by households in steady state.

Finally, setting $\overline{r} = \rho - \sigma$ in (a.17), we can write the IS equation as

$$\dot{x}_t = (\phi_\pi - 1)\pi_t^g + \sigma\left(e^{-\gamma\Theta x_t} - 1\right),$$

which is the same as (18a) in the main text. □

B Baseline model with inflation targeting rule

This section contains the proof of claims relating to the baseline model described by (18a)-(18b).

B.1 Global determinacy in RANK and in HANK with acyclical risk

In the RANK limit ($\sigma = 0$)/ with acyclical risk ($\sigma > 0, \Theta = 0$), aggregate dynamics are given by:

$$\dot{x}_t = (\phi_\pi - 1)\pi_t^g$$
$$\dot{\pi}_t^g = \rho\pi_t^g - \kappa(e^{x_t} - 1)$$

The Jacobian of the system evaluated at any (x, π^g) can be written as:

$$\begin{bmatrix} 0 & \phi_\pi - 1 \\ -\kappa e^x & \rho \end{bmatrix}$$

To show that $\phi_\pi > 1$ delivers global determinacy, we can invoke the Bendixson–Dulac theorem (Bendixson, 1901; Dulac, 1937),[43] which states if the trace does not change sign anywhere in the domain, then there are no non-constant periodic solutions lying entirely within $(x, \pi^g) \in (-\infty, \infty)^2$. This is true by inspection since the trace is given by $\rho > 0$. Thus, there are no non-constant periodic solutions.

Next, with $\phi_\pi > 1$, the determinant of the Jabobian evaluated at the targeted steady state $(x, \pi^g) = (0, 0)$ is given by $\kappa(\phi_\pi - 1) > 0$. Together with the fact that the trace is always positive $\rho > 0$, this implies that both eigenvalues have positive real parts. Thus, the targeted steady state $(0, 0)$ is unstable, and hence the only bounded equilibrium is given by the trajectory $(x_t, \pi_t^g) = (0, 0)$ for all t. □

B.2 Proof of Proposition 2

Close to the targeted steady state $(0, 0)$, the dynamics of the system (18a)-(18b) are governed by:

$$\begin{bmatrix} \dot{x} \\ \dot{\pi}_t^g \end{bmatrix} = A \begin{bmatrix} x \\ \pi^g \end{bmatrix} + \mathcal{O}\left(x^2\right) \qquad \text{for} \qquad (x, \pi) \to (0, 0),$$

where A is given by

$$A = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

Since both x and π^g are "jump" variables, local determinacy requires that both eigenvalues of A have a positive real part. As is well known, the sum of the two eigenvalues of A, denoted by z_1 and z_2, is given by the trace of A, while their product is given by the determinant of A:

$$\begin{aligned} z_1 + z_2 &= \rho - \sigma\gamma\Theta, \\ z_1 \times z_2 &= \kappa(\phi_\pi - 1) - \rho\sigma\gamma\Theta \end{aligned}$$

Since this is a two dimensional system, either both z_1 and z_2 are real, or they are complex conjugates. Thus, for z_1 and z_2 to both have positive real parts, it is sufficient that both the sum and product of z_1, z_2 be positive. In other words, as long as $\Theta < \Theta^* \equiv \frac{\rho}{\sigma\gamma}$, a sufficient condition for local determinacy is that

$$\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa},$$

which is the same condition as in Proposition 2. Finally, for $\Theta > \Theta^*$, the sum of the two eigenvalues $z_1 + z_2 < 0$ regardless of the magnitude of ϕ_π, implying that at least one of the eigenvalues must have a negative real part, i.e., regardless of the magnitude of ϕ_π, the equilibrium is locally indeterminate. □

[43] See Theorem 4.1 on page 39 in Verhulst (1990) for a simple statement of the Bendixson-Dulac theorem in English.

B.3 Multiple Steady States

For any $\Theta > 0$, our baseline HANK economy has two steady states (except in a knife edge case). The $\dot{x} = 0$ and $\dot{\pi}_t^g = 0$ nullclines imply that in any steady state, (x, π) must satisfy:

$$\begin{aligned} 0 &= (\phi_\pi - 1)\,\pi^g + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right) \\ 0 &= \rho\pi^g - \kappa\,(\mathrm{e}^x - 1) \end{aligned}$$

Clearly, $(0, 0)$ always satisfies both equations. To see that there is generically another steady state, combine the two equations to eliminate π^g, to get an expression exclusively in terms of x:

$$F(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}(\mathrm{e}^x - 1) + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right), \tag{b.1}$$

and any x which satisfies $F(x) = 0$ constitutes a steady state. Again, clearly $x = 0$ solves this equation. The derivative of $F(x)$ is given by:

$$F'(x) = \frac{\kappa\,(\phi_\pi - 1)}{\rho}\mathrm{e}^x - \sigma\gamma\Theta\mathrm{e}^{-\gamma\Theta x},$$

which, evaluated at $x = 0$ yields

$$F'(0) = \frac{\kappa}{\rho}\Big(\phi_\pi - \varphi(\Theta)\Big) \qquad \text{where} \qquad \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa},$$

If $\phi_\pi = \varphi(\Theta)$, then $F'(0) = 0$ and $F(x)$ is tangent to the x-axis at $x = 0$, implying that it is the only zero of $F(x)$ since $F(x)$ is declining in the region $x = 0$ and increasing in the region $x > 0$. This is the knife edge case in which there is a unique steady state. If instead, $\phi_\pi > \varphi(\Theta)$, then $F'(0) > 0$. Since $\lim_{x\to-\infty} F(x) \to \infty$, there must be at least one intersection with $\underline{x} < 0$ and $F'(\underline{x}) < 0$. Since $F(x)$ is strictly convex, this intersection is unique. Further, note that $dF(x)/d\phi_\pi < 0$ for $x < 0$ by inspection. Thus, by the implicit function theorem, we have $d\underline{x}/d\phi_\pi < 0$.

Instead if $\phi_\pi < \varphi(\Theta)$, then $F(x)$ intersects the x axis twice, one of which is $x = 0$. We also know that in this case $F'(0) < 0$ and that $F(x) \to \infty$ as $x \to \infty$, implying that there is at least one intersection at $\overline{x} > 0$ with $F'(\overline{x}) > 0$. Since $F(x)$ is convex as long as $\phi_\pi > 1$, this intersection is the only other intersection except $x = 0$. Furthermore, the implicit function theorem implies that a smaller ϕ_π implies a larger $\overline{x}$ as long as $\phi_\pi > 1$. □

B.4 Local stability of the untargeted steady state

For any $\Theta > 0$, we focus of the case in which $\phi_\pi > \varphi(\Theta)$, which is a necessary (but not sufficient) condition for the targeted equilibrium to be locally determinate (see Proposition 2). We now show that whenever this condition is satisfied, the untargeted steady state, which features lower output $\underline{x}$. At the untargeted steady state, the Jacobian of the system (18a)-(18b) can be written as:

$$A_{\underline{x}} = \begin{bmatrix} -\sigma\gamma\Theta e^{-\gamma\Theta\underline{x}} & \phi_\pi - 1 \\ -\kappa e^{\underline{x}} & \rho \end{bmatrix} \tag{b.2}$$

Thus, up to first-order, the dynamics around the untargeted steady state are identical to the local dynamics around the *targeted* steady state of an alternate economy with more cyclical risk: $\Theta' = \Theta e^{-\gamma\Theta \underline{x}} > \Theta$ and a flatter Phillips curve with slope $\kappa' = \kappa e^{\underline{x}} < \kappa$ (since $\underline{x} < 0$, and so $e^{-\gamma\Theta\underline{x}} > 1$ and $e^{\underline{x}} < 1$). Consequently, we can apply Proposition 2 to conclude that for the untargeted steady state $(\underline{x}, \underline{\pi}^g)$ to be unstable (locally determinate), we need:

$$\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta'}{\kappa'} = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(\gamma\Theta+1)\underline{x}}$$

However, for a given Θ and ϕ_π, this can never be satisfied as long as $\phi_\pi > \varphi(\Theta)$. To see why, recall from Appendix B.3 that we can use (b.1) to write $d\underline{x}/d\phi_\pi$ as:

$$\frac{d\underline{x}}{d\phi_\pi} = \left(\frac{e^{\underline{x}}}{1-e^{\underline{x}}}\right)\left[\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}\right]$$

We know that this expression is negative as long as $\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$. Since $\underline{x} < 0$, this implies that

$$\phi_\pi < 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}, \tag{b.3}$$

i.e., the untargeted steady state $(\underline{x}, \underline{\pi}^g)$ is stable (locally indeterminate), if $\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$. □

B.5 Proof of Proposition 3

Given the interest rate rule $i_t = \bar{r} + \phi_\pi \pi_t^g$, the aggregate dynamics of the output-gap x_t and inflation-gap π_t^g is given by the following 2 dimensional system of ordinary differential equations:

$$\begin{aligned} \dot{x} &= (\phi_\pi - 1)\,\pi^g + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\ \dot{\pi} &= \rho\pi - \kappa\,(e^x - 1) \end{aligned}$$

We can rewrite this system in matrix form as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \end{bmatrix} = \underbrace{\begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \pi_t^g \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\gamma\Theta x_t} - 1 + \gamma\Theta x_t\right) \\ -\kappa\left(e^x - 1 - x_t\right) \end{bmatrix} \tag{b.4}$$

We will prove Proposition 3 by using Theorem 7.1 in Kopell and Howard (1975), and the Hopf bifurcation theorem (Marsden and McCracken, 1976). We present these theorems here for convenience.

Theorem 1 (Hopf Bifurcation Theorem)**.** *Consider a two-dimensional system*

$$\dot{\mathbf{x}} = F_\mu(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2, \mu \in \mathbb{R}$$

with smooth F, which for all sufficiently small $|\mu|$ has the equilibrium $\mathbf{x} = (0,0)$, and the Jacobian $D_{\mathbf{x}}F_\mu(0,0)$ has eigenvalues

$$\lambda_{1,2}(\mu) = \Omega(\mu) \pm \mathtt{i}\omega(\mu) \qquad \textit{where} \qquad \mathtt{i} = \sqrt{-1},$$

Then, if the following conditions are satisfied:

1. *At $\mu = 0$, there exists a purely imaginary set of eigenvalues: $\Omega(0) = 0$ and $\omega(0) > 0$*
2. *The eigenvalues cross the imaginary axis with non-zero speed: $\frac{d\Omega(0)}{d\mu} \neq 0$*
3. *The first Lyapunov coefficient of the system $\ell_1(0) \neq 0$,*

there exists a family of real periodic solutions $\mathbf{x} = \mathbf{x}(t, \epsilon)$, $\mu(\epsilon)$ which has properties $\mu(0) = 0$, $\mathbf{x}(t, 0) = (0, 0)$, but $\mathbf{x}(t, \epsilon) \neq (0, 0)$ for sufficiently small ϵ. The same holds for the period $T(\epsilon)$ and $T(0) = 2\pi / |\omega(0)|$.

Theorem 2 (Theorem 7.1 in Kopell and Howard (1975)). *Let $\dot{\mathbf{x}} = F_{\mu,\nu}(\mathbf{x})$ be a two parameter family of ODEs on $\mathbb{R}^2$, F smooth in all of its four arguments, such that $F_{\mu,\nu}(0,0) = \mathbf{0}$. Also assume:*

1. *$dF_{0,0}(0,0)$ has a double zero eigenvalue and a single eigenvector $\mathbf{e}$.*
2. *The mapping $(\mu, \nu) \to (det(dF_{0,0}(0,0), tr(dF_{0,0}(0,0))$ has a nonzero Jacobian at $(\mu, \nu) = (0, 0)$.*
3. *Let $Q(\mathbf{x}, \mathbf{x})$ be the 2×1 vector containing the terms quadratic in $\mathbf{x}$ and independent of (μ, ν) in a Taylor series expansion of $F_{\mu,\nu}(\mathbf{x})$ around 0. Then $[dF_{0,0}(0,0), Q(\mathbf{e}, \mathbf{e})]$ has rank 2.*

Then there is a curve $f(\mu, \nu) = 0$ such that if $f(\mu_0, \nu_0) = 0$, then $\dot{\mathbf{x}} = F_{\mu_0,\nu_0}(\mathbf{x})$ has a homoclinic orbit. This one-parameter family of homoclinic orbits (in (X, μ, ν) space) is on the boundary of a two-parameter family of periodic solutions. For all $|\mu|, |\nu|$ sufficiently small, if $\dot{\mathbf{x}} = F_{\mu,\nu}(\mathbf{x})$ has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.

We first use Theorem 1 to prove point 2 of Proposition 3. We use the cyclicality of risk Θ as the bifurcation parameter (which plays the role of μ in Theorem 1 above). Define $\Theta^\star = \frac{\rho}{\sigma\gamma}$. Imposing $\Theta = \Theta^\star$ in (b.4) yields

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}^g_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\rho & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A^\star} \begin{bmatrix} x_t \\ \pi^g_t \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\frac{\rho}{\sigma}x_t} - 1 + \frac{\rho}{\sigma}x_t\right) \\ -\kappa\left(e^{x} - 1 - x_t\right) \end{bmatrix}, \tag{b.5}$$

It is clear by inspection that the trace of the matrix $A^\star$ is equal to zero, which implies that the two eigenvalues sum to 0. Also, the determinant of $A^\star$ is given by

$$\text{Det}(A^\star) = \kappa(\phi_\pi - 1) - \rho^2$$

Next, recall that the augmented Taylor principle (19) requires that $\phi_\pi > 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$ for local determinacy. Evaluating this expression at $\Theta = \Theta^\star$, we have $\phi_\pi > 1 + \frac{\rho^2}{\kappa}$, which in turn implies that $\text{Det}(A^\star) > 0$. Consequently, at $\Theta = \Theta^\star$ the eigenvalues of $A^\star$ are purely imaginary and given by $\pm\omega\mathtt{i}$, where $\mathtt{i} = \sqrt{-1}$ and $\omega = \sqrt{\kappa(\phi_\pi - 1) - \rho^2}$. Thus, requirement 1 of Theorem 1 is satisfied at $\Theta = \Theta^\star$ (this corresponds to the $\mu = 0$ in the statement of the theorem).

Next, it is clear by inspection that the eigenvalues of the matrix A in (b.4) change smoothly in Θ, which implies that condition 2 of Theorem 1 is also satisfied. Thus, the only other condition we need to check is that the first Lyapunov coefficient (evaluated at the bifurcation point) is not 0. To check this, we first need to transform the system (b.5) into normal-form, for which we diagonalize $A^{\star}$ as

$$A^{\star} = PDP^{-1},$$

where

$$D = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \rho & \omega \\ \kappa & 0 \end{bmatrix}$$

Next, we can pre-multiply both sides of (b.5) by P^{-1} to express the system in normal form:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}, \tag{b.6}$$

where

$$\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ \pi^{g} \end{bmatrix},$$

$$\begin{aligned} f(u,v) &= -e^{\rho u + \omega v} + 1 + \rho u + \omega v \\ g(u,v) &= \frac{\rho e^{\rho u + \omega v} + \sigma e^{-\frac{\rho}{\sigma}(\rho u + \omega v)} - (\rho + \sigma)}{\omega} \end{aligned}$$

Finally, the first Lyapunov coefficient at the bifurcation point $\Theta = \Theta^{\star}$ is given by:

$$\begin{aligned} \ell_1(0) &= f_{uuu}(0,0) + f_{uvv}(0,0) + g_{uuv}(0,0) + g_{vvv}(0,0) \\ &\quad + \frac{1}{\omega}\Big[f_{uv}(0,0)\left(f_{uu}(0,0) + f_{vv}(0,0)\right) - g_{uv}(0,0)\left(g_{uu}(0,0) + g_{vv}(0,0)\right) - f_{uu}(0,0)\, g_{uu}(0,0) \\ &\quad + f_{vv}(0,0)\, g_{vv}(0,0) \Big] \\ &= -(\rho + \sigma)\frac{\rho^2 \kappa^2 (\phi_{\pi} - 1)^2}{\sigma^2 \omega^2} < 0, \end{aligned}$$

which is non-zero, for any $\phi_{\pi} > 0$. Thus, condition 3 of Theorem 1 is also satisfied, and the system (b.4) undergoes a Hopf bifurcation at $\Theta = \Theta^{\star}$. Furthermore, since $\ell_1(0)$ regardless of the value of $\phi_{\pi} > 1$, the Hopf bifurcation is always *supercritical*, i.e. the higher order terms of the system (b.4), push $\mathbf{x}$ in towards the equilibrium $(0,0)$.

In terms of our HANK model, this means that for $\Theta < \Theta^{\star}$ in the neighborhood of $\Theta^{\star}$, starting from any initial condition in the neighborhood $(x, \pi^{g}) = (0,0)$, the system converges to a stable cycle. Thus, for any Θ in this neighborhood, even though the equilibrium is locally determinate, the equilibrium is globally indeterminate since all trajectories starting near the targeted steady state initially diverge but then converge to a cycle, remaining bounded. As point 2 of Proposition 3 states, the system converges to a cycle as long as $\Theta \in (\overline{\Theta}, \Theta^{\star})$, where holding all other parameters fixed, $\overline{\Theta}$ is implicitly given by the value of Θ for which $\omega(\overline{\Theta}) = 0$.

Next, we characterize what happens at the boundaries of this neighborhood. First, lets consider the case $\Theta = \Theta^{\star}$. Since the eigenvalues of the Jacobian at $\Theta = \Theta^{\star}$ are purely imaginary, the first-order terms do not move the system away from or towards $(0,0)$. However, since the Hopf bifurcation is supercritical, the higher-order terms push in towards the origin, implying that all trajectories which start in the neighborhood of the targeted steady state $(0,0)$ remain bounded, implying global indeterminacy.

If instead $\Theta = \overline{\Theta}$, Proposition 3 states that there exists a homoclinic orbit. To prove this claim, we need to use Theorem 2. Theorem 2 states that as long the conditions 1,2,3 are satisfied, then (b.4) has a homoclinic orbit on the boundary of the stable cycles we described above. While Theorem 1 only required conditions on one parameter Θ, Theorem 2 requires imposing some additional conditions on a second parameter. For us, it is most convenient to choose ϕ_π as the second parameter. First, notice that for $\Theta = \Theta^\star$ and $\phi_\pi = \varphi(\Theta^\star) = 1 + \frac{\rho^2}{\kappa}$, the Jacobian of (b.4) is:

$$A^\diamond = \begin{bmatrix} -\rho & \frac{\rho^2}{\kappa} \\ -\kappa & \rho \end{bmatrix}, \tag{b.7}$$

which has both trace and determinant equal to 0, implying that both eigenvalues are 0, satisfying condition 1 of Theorem 2. Also, since the eigenvalues repeat, it is easy to check that the matrix has eigenvector $\mathbf{e} = \begin{bmatrix} \rho \\ \kappa \end{bmatrix}$, and a generalized eigenvector $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Next, we show that condition 2 is also satisfied. Recall that the Jacobian of (b.4) is given by the matrix A

$$A(\Theta, \phi_\pi) = \begin{bmatrix} -\sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

and the trace of $A(\Theta, \phi_\pi)$ is $Tr_A = \rho - \sigma\gamma\Theta$, while the determinant is $Det_A = \kappa(\phi_\pi - 1) - \rho\sigma\gamma\Theta$. Then the Jacobian of $[Tr_A, Det_A]$ is given by:

$$J = \begin{bmatrix} -\sigma\gamma & -\rho\sigma\gamma \\ 0 & \kappa \end{bmatrix}$$

The determinant of this matrix is non-zero as long as $\kappa\sigma\gamma \neq 0$. This confirms that condition 2 of Theorem 2 is satisfied. Next, to check condition 3, we need to construct the matrix $Q(\mathbf{x}, \mathbf{x})$, which is a 2×1 vector which contains terms quadratic in (x, π^g) and independent of (Θ, ϕ_π) in a Taylor series expansion of $F\Big(x, \pi^g, \Theta^\star, \varphi(\Theta^\star)\Big)$ around $(x, \pi^g) = (0,0)$. Since (b.4) has no higher-order terms in π^g, it is easy to see that $Q(\mathbf{x}, \mathbf{x})$ can be written as:

$$Q = \begin{bmatrix} 0 \\ -0.5\kappa x^2 \end{bmatrix}$$

Evaluating this at the eigenvector $\mathbf{e}$, we have:

$$Q(\mathbf{e}, \mathbf{e}) = \begin{bmatrix} 0 \\ -0.5\kappa\rho^2 \end{bmatrix}$$

Then, clearly the condition 3 of Theorem 2 is satisfied since

$$\text{rank}\begin{bmatrix} -\rho & \kappa^{-1}\rho^2 & 0 \\ -\kappa & \rho & -0.5\kappa\rho^2 \end{bmatrix} = 2,$$

as long as $\rho \neq 0$. Thus, the conditions for Theorem 2 are satisfied. Consequently, Theorem 2 states that at $\Theta = \overline{\Theta}$, a homoclinic orbit emerges, which completes the proof of point 2 of Proposition 3.

Next, we prove point 1 of Proposition 3. Point 2 of Proposition 3 established that stable cycles exist for $\Theta \in (\overline{\Theta}, \Theta^{\star})$ and a homoclinic orbit exists at $\Theta = \overline{\Theta}$. Recall that for $\Theta < \overline{\Theta}$, the roots are no longer complex since $\omega(\overline{\Theta}) = 0$, and there are no cycles. Then Theorem 2 implies that since there is no cycle or homoclinic orbit, there exists a saddle connection along which the economy moves from the targeted to the untargeted steady state for $0 < \Theta < \overline{\Theta}$. Any trajectory which originates on this saddle connection remain bounded. Hence, there is global indeterminacy even when $0 < \Theta < \overline{\Theta}$.

Finally, point 3 of Proposition 3 is also true because of similar reasons. When $\Theta > \Theta^{\star}$, the Hopf bifurcation theorem implies that there are no cycles in this part of the parameter space. Consequently, Theorem 2 implies that there must be a saddle connection from the untargeted to targeted steady state. Since the targeted equilibrium is already locally indeterminate for $\Theta > \Theta^{\star}$, there is also global indeterminacy. This concludes the proof of Proposition 3. □

C Monetary policy rule with output-gap stabilization

This section contains the proof of claims relating to the model described by (23a)-(23b).

C.1 Proof of Proposition 4

With $\phi_x \neq 0$, for (x, π^g) close to the targeted steady state $(0,0)$, the dynamics of the system (18a)-(18b) are governed by the following system:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t^g \end{bmatrix} + \mathcal{O}\left(x^2\right) \qquad \text{for}(x,\pi) \to (0,0),$$

where A is given by

$$A = \begin{bmatrix} \phi_x - \sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

Since both x and π^g are jump-variables, local determinacy requires that both eigenvalues of A have a positive real part. As is well known, the sum of the two eigenvalues of A, denoted by z_1 and z_2, is given by the trace of A, while their product is given by the determinant of A:

$$\begin{aligned} z_1 + z_2 &= \rho + \phi_x - \sigma\gamma\Theta, \\ z_1 \times z_2 &= \kappa(\phi_\pi - 1) + \rho\phi_x - \rho\sigma\gamma\Theta \end{aligned}$$

Since this is a two dimensional system, either both z_1 and z_2 are real, or they are complex conjugates. Thus, for z_1 and z_2 to both have positive real parts, it is sufficient that both the sum and product of z_1, z_2 be positive. In other words, a sufficient condition for local determinacy is

$$\phi_\pi + \frac{\rho}{\kappa}\phi_x > \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} \qquad \text{and} \qquad \phi_x > \sigma\gamma\left(\Theta - \Theta^\star\right) \tag{c.1}$$

This condition is satisfied if (ϕ_π, ϕ_x) jointly satisfy the following condition

$$\phi_\pi > \varphi(\Theta) \qquad \text{and} \qquad \phi_x > \sigma\gamma\left(\Theta - \Theta^\star\right),$$

which is the same condition as in Proposition 4. For mildly or moderately countercyclical risk $0 < \Theta < \Theta^\star$, the above expression shows that setting $\phi_x = 0$ and $\phi_\pi > \varphi(\Theta)$ is sufficient for local determinacy. This is the same condition as in Proposition 2. However, when risk is highly countercyclical $\Theta \geq \Theta^\star$, setting $\phi_x = 0$ is no longer enough for local determinacy. Local determinacy now requires a large enough $\phi_x > \sigma\gamma(\Theta - \Theta^\star)$ alongside $\phi_\pi > \varphi(\Theta)$. □

C.2 Multiple Steady States

Even with $\phi_x > 0$, the untargeted steady state survives as long as $\Theta > 0$. The $\dot{x} = 0$ and $\dot{\pi}^g_t = 0$ nullclines, in this case, imply that in any steady state, (x, π^g) must satisfy:

$$\begin{aligned} 0 &= \left(\phi_\pi - 1\right)\pi^g + \phi_x x + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\ 0 &= \rho\pi^g - \kappa\left(e^x - 1\right) \end{aligned}$$

Clearly, $(0, 0)$ still satisfies both equations. To see that the untargeted steady state still exists, combine the two equations to eliminate π^g, to get an expression exclusively in terms of x:

$$F(x) = \frac{\kappa\left(\phi_\pi - 1\right)}{\rho}\left(e^x - 1\right) + \phi_x x + \sigma\left(e^{-\gamma\Theta x} - 1\right),$$

and any x which solves $F(x) = 0$ constitutes a steady state. Again, clearly $x = 0$ solves this equation. The derivative of $F(x)$ is given by:

$$F'(x) = \frac{\kappa\left(\phi_\pi - 1\right)}{\rho}e^x + \phi_x - \sigma\gamma\Theta e^{-\gamma\Theta x},$$

which, evaluated at $x = 0$ yields

$$F'(0) = \frac{\kappa}{\rho}\left(\phi_\pi + \frac{\rho}{\kappa}\phi_x - \varphi\left(\Theta\right)\right) \qquad \text{where} \qquad \varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa},$$

If $\phi_\pi + \frac{\rho}{\kappa} = \varphi(\Theta)$, then $F'(0) = 0$ and $F(x)$ is tangent to the x-axis at $x = 0$, implying that it its the only zero of $F(x)$ since $F(x)$ is declining in the region $x = 0$ and increasing in the region $x > 0$. This is the knife edge case in which there is a unique steady state. If instead, we impose the condition for local determinacy of the targeted equilibrium, $\phi_\pi + \frac{\rho}{\kappa}\phi_x > \varphi(\Theta)$, then $F'(0) > 0$. Since $\lim_{x\to-\infty} F(x) \to \infty$, there must be at least one intersection with $\underline{x} < 0$ and $F'(\underline{x}) < 0$. Since $F(x)$ is strictly convex, this intersection is unique. Further, note that $dF(x)/d\phi_\pi < 0$ for $x < 0$ by inspection. Thus, by the implicit function theorem, we have $d\underline{x}/d\phi_\pi < 0$. $\square$

C.3 Global indeterminacy

Given the interest rate rule $i_t = \bar{r} + \phi_\pi \pi_t^g + \phi_x x_t$, the dynamics of x_t and π_t^g are given by the ODEs:

$$\begin{aligned} \dot{x} &= (\phi_\pi - 1)\,\pi^g + \phi_x x_t + \sigma\left(e^{-\gamma\Theta x} - 1\right) \\ \dot{\pi} &= \rho\pi - \kappa\,(e^x - 1) \end{aligned}$$

We can rewrite this system in matrix form as:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_x - \sigma\gamma\Theta & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \pi_t^g \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\gamma\Theta x_t} - 1 + \gamma\Theta x_t\right) \\ -\kappa\,(e^x - 1 - x_t) \end{bmatrix} \tag{c.2}$$

Notice that the matrix A has has trace equal to zero at $\phi_x = \phi_x^\star = \sigma\gamma\Theta - \rho = \sigma\gamma(\Theta - \Theta^\star)$, where $\Theta^\star = \frac{\rho}{\sigma\gamma}$. Evaluating (c.2) at $\phi_x = \phi_x^\star$, we have:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \end{bmatrix} = \underbrace{\begin{bmatrix} -\rho & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A^\star} \begin{bmatrix} x_t \\ \pi_t^g \end{bmatrix} + \begin{bmatrix} \sigma\left(e^{-\frac{\rho}{\sigma}x_t} - 1 + \frac{\rho}{\sigma}x_t\right) \\ -\kappa\,(e^x - 1 - x_t) \end{bmatrix}, \tag{c.3}$$

which is identical to (b.5) in the model with $\phi_x = 0$ (See Appendix B.5 for details). Consequently, all the conditions for Theorem 1 are satisfied, and the system undergoes a Hopf bifurcation at $\phi_x = \phi_x^\star$. Consequently, $\exists\overline{\phi}_x > \phi_x^\star$ such that for $\phi_x \in (\phi_x^\star, \overline{\phi}_x)$, trajectories starting in the neighborhood of $(0,0)$ initially diverge, but then converge to a stable cycle which surrounds the targeted steady state. Since all these trajectories remain bounded, in this region $\phi_x \in (\phi_x^\star, \overline{\phi}_x)$, the equilibrium is globally indeterminate, even though the targeted steady state is unstable (locally determinate). For $\phi_x = \phi_x^\star$, the stable cycles collapse onto $(0,0)$. The equilibrium in this case is still globally indeterminate because the higher-order terms of the system push any trajectory originating near the targeted steady state towards $(0,0)$. Next, evaluating $A^\star$ in (c.3) at $\phi_\pi = \varphi(\Theta)$ yields the same matrix as $A^\diamond$ in (b.7) in Appendix B.5. Furthermore, since setting $\phi_x \neq 0$ does not change the higher-order terms of the system, by the same reasoning as in Appendix B.5, all the conditions of Theorem 2 are also satisfied with $\phi_x \neq 0$. Thus, it follows that for $\phi_x = \overline{\phi}_x$, the stable cycles get absorbed into a homoclinic orbit. This proves point 2 of Proposition 5.

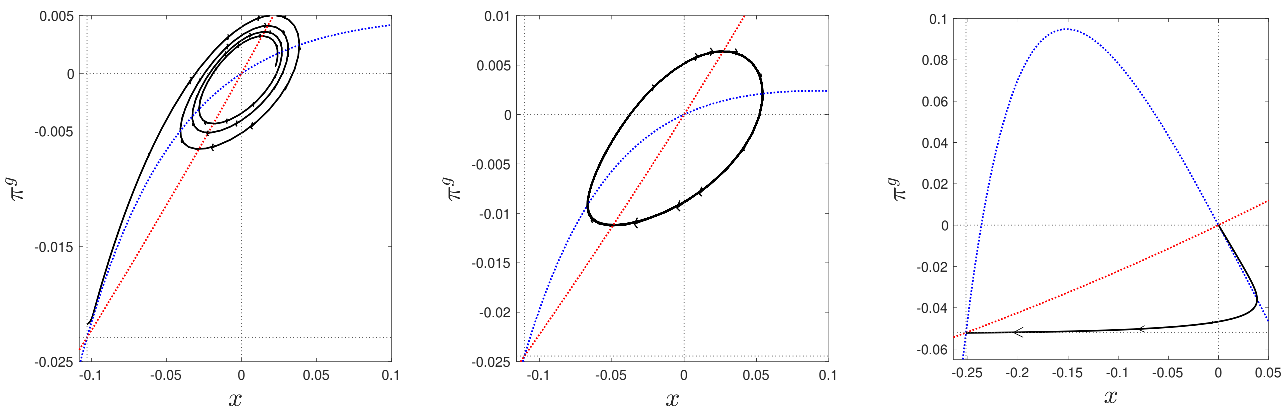
For $\phi_x > \overline{\phi}_x$, while the stable cycles disappear but the equilibrium is still globally indeterminate. This is because Theorem 2 guarantees the existence of a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. All trajectories beginning from any point on this saddle connection always remain bounded, thus proving the existence of multiple bounded trajectories which satisfy all equilibrium conditions. Thus even though for a large ϕ_x, the targeted equilibrium is locally determinate, there is global indeterminacy.

This proves point 3 of Proposition 5.

For $\phi_x < \phi_x^\star$, Proposition 4 proves that the targeted equilibrium is locally indeterminate. Thus, the equilibrium is also globally indeterminate. In addition, Theorem 2 ensures the existence of a saddle connection along which the economy can converge from the neighborhood of the untargeted steady state to the targeted steady state. This proves point 1 of Proposition 5.

□

C.4 Plot depicting global dynamics with highly countercyclical risk



(a) small ϕ_x (b) medium ϕ_x (c) large ϕ_x

Figure 7: Global dynamics as a function of ϕ_x when $\Theta > \Theta^\star$

Figure 7 depicts global dynamics with a policy rule which puts some weight on stabilizing the output-gap. This Figure considers the case with highly countercyclical risk ($\Theta > \Theta^\star$). In particular, we set $\Theta = 32$, which given our calibration is larger than $\Theta^\star = 31.1$. We impose $\phi_\pi > \varphi(\Theta)$ in all plots. Figure 7a plots dynamics when $\phi_x < \sigma\gamma(\Theta - \Theta^\star)$. As stated in Propositions 4 and 5, in this region of ϕ_x, the equilibrium is both locally and globally indeterminate. The Figure shows a saddle connection along which the economy moves from the neighborhood of the untargeted steady state to the targeted steady state. Figure 7b plots dynamics in the region where $\phi_x \in [\phi_x^\star, \overline{\phi}_x]$. The black trajectory depicts a stable cycle. All trajectories originating near the targeted steady state initially diverge away from it but then converge to the stable cycle and remain bounded. Thus, while the targeted equilibrium is locally determinate, it is globally indeterminate. Finally, Figure 7c plots global dynamics in the case with $\phi_x > \overline{\phi}_x$. Propositions 4 and 5 show that in this region the targeted equilibrium is locally determinate but globally indeterminate. The black trajectory depicts a saddle connection along which a trajectory starting near the targeted steady state diverges away from it, only to converge to the untargeted steady state. Thus, Figures 5 and 7 show that for any $\Theta > 0$, the equilibrium is always globally indeterminate, regardless of the combination of ϕ_π, ϕ_x.

D Inertial policy rule

This section contains the proof of claims relating to the model described by (27), (28) and (29).

D.1 Local stability of the targeted steady state

With the inertial rule $\alpha > 0$, for (x, π^g, π^b) close to the targeted steady state $(0,0,0)$, the dynamics of the system (27), (28) and (29) are governed by the following system:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \\ \dot{\pi}_t^b \end{bmatrix} = A \begin{bmatrix} x_t \\ \pi_t^g \\ \pi_t^b \end{bmatrix} + \mathcal{O}\left(x^2\right) \qquad \text{for}(x, \pi, \pi^b) \to (0, 0, \pi^\star),$$

where A is given by

$$A = \begin{bmatrix} -\sigma\gamma\Theta & -1 & \phi_\pi \\ -\kappa & \rho & 0 \\ 0 & \alpha & -\alpha \end{bmatrix},$$

Since we have one predetermined-variable π^b and two jump-variables x, π^g, for the targeted steady state to be locally determinate, we need one negative root and two roots with positive real parts. This would ensure that for a given π_0^b in the neighborhood of the targeted steady state, there exists a unique (x_0, π_0^g) such that the trajectory $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ remains bounded.

To see that this is the case when ϕ_π is large enough and α is small enough, we need to characterize the eigenvalues of A. The characteristic polynomial associated with A can be written as:

$$\mathcal{P}(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

where

$$\begin{aligned} a_0 &= -1 \\ a_1 &= -\sigma\gamma\left(\Theta - \Theta^\star\right) - \alpha \\ a_2 &= \kappa\varphi(\Theta) - \alpha\sigma\gamma\left(\Theta - \Theta^\star\right) \\ a_3 &= -\alpha\kappa\Big(\phi_\pi - \varphi(\Theta)\Big), \end{aligned}$$

where

$$\varphi(\Theta) = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa}$$

The stability of the system is governed by the pattern of sign changes in the sequence:

$$a_0, \qquad a_1, \qquad -\frac{a_0 a_3 - a_1 a_2}{a_1}, \qquad a_3$$

For the Jacobian to have one negative root and two roots with positive real parts, we need the sequence to have 2 sign changes. The first term in this sequence is always $-$. Imposing $\phi_\pi > \varphi(\Theta)$ guarantees that the fourth term in the sequence is also $-$. To determine the sign of the other two terms in the sequence, we need to consider two cases:

(i) **Highly countercyclical risk** $(\Theta > \Theta^\star)$: In this case, for any $\alpha > 0$, the second term in the sequence a_1 is negative. So the sequence is $-,-,?,-$. Local determinacy then requires that the third term in the sequence be positive (two sign changes) for local determinacy, i.e,

$$\frac{\sigma\gamma\left(\Theta-\Theta^{\star}\right)\left[\kappa\varphi\left(\Theta\right)-\alpha\sigma\gamma\left(\Theta-\Theta^{\star}\right)-\alpha^{2}\right]+\alpha\kappa\phi_{\pi}}{\sigma\gamma\left(\Theta-\Theta^{\star}\right)+\alpha}>0,$$

which can be reformulated as:

$$\psi\left(\alpha\right)<\phi_{\pi},$$

where

$$\psi(\alpha)=-\frac{\sigma\gamma\left(\Theta-\Theta^{\star}\right)}{\alpha\kappa}\left[\kappa\varphi\left(\Theta\right)-\alpha\sigma\gamma\left(\Theta-\Theta^{\star}\right)-\alpha^{2}\right] \tag{d.1}$$

We know that

$$\psi'\left(\alpha\right)=\frac{\sigma\gamma\left(\Theta-\Theta^{\star}\right)}{\kappa}\left[1+\frac{\kappa\varphi\left(\Theta\right)}{\alpha^{2}}\right]>0 \qquad \psi\left(0\right)\rightarrow-\infty \qquad \psi\left(\infty\right)\rightarrow\infty,$$

which, by the intermediate-value theorem implies that $\exists\alpha^{\star}(\Theta)\in(0,\infty)$, such that $\psi\Big(\alpha^{\star}(\Theta)\Big)=\phi_{\pi}$ and $\psi(\alpha)<\phi_{\pi}$ for all $\alpha\in\Big(0,\alpha^{\star}(\Theta)\Big)$. In fact, we can write $\alpha^{\star}(\Theta)$ as:

$$\alpha^{\star}(\Theta)=\frac{1}{2}\left\{\frac{\kappa\phi_{\pi}-\sigma^{2}\gamma^{2}\left(\Theta-\Theta^{\star}\right)^{2}}{\sigma\gamma\left(\Theta-\Theta^{\star}\right)}+\sqrt{\frac{\left[\kappa\phi_{\pi}-\sigma^{2}\gamma^{2}\left(\Theta-\Theta^{\star}\right)^{2}\right]^{2}}{\sigma^{2}\gamma^{2}\left(\Theta-\Theta^{\star}\right)^{2}}+4\kappa\varphi\left(\Theta\right)}\right\}$$

Thus, even when risk is highly countercyclical $\Theta>\Theta^{\star}$ local determinacy is ensured as long as α is small enough, i.e., the rule is backward-looking enough.

(ii) **Mildly or moderately countercyclical risk** $(\Theta\leq\Theta^{\star})$: In this region, we need to check two cases. First consider the case in which α is large: $\alpha\geq\sigma\gamma(\Theta^{\star}-\Theta)$. In this case, the second term is still negative, and so the sequence is $-,-,?,-$. Thus, we need the third term in the sequence to be positive. For this to be the case, we need

$$\phi_{\pi}>\psi\left(\alpha\right),$$

where $\psi(\alpha)$ is the same as in (d.1). However, now with $\Theta<\Theta^{\star}$, we have:

$$\psi'\left(\alpha\right)=\frac{\sigma\gamma\left(\Theta-\Theta^{\star}\right)}{\kappa}\left[1+\frac{\kappa\varphi\left(\Theta\right)}{\alpha^{2}}\right]<0 \qquad \psi\Big(\sigma\gamma(\Theta^{\star}-\Theta)\Big)=\varphi(\Theta) \qquad \psi\left(\infty\right)\rightarrow-\infty,$$

Thus, the third term is always positive in this case. $\alpha\geq\sigma\gamma(\Theta^{\star}-\Theta)$. Thus, we have local determinacy for any α in this region.

Finally, the remaining case is when α is small: $0<\alpha<\sigma\gamma(\Theta^{\star}-\Theta)$. In this case, the second term of the sequence is positive. So the sequence is $-,+,?,-$. Thus, there are two sign changes regardless of the sign of the third term, and we have local determinacy for any α in this region.

Overall, if $\Theta \in (0, \Theta^\star]$, the targeted equilibrium is locally determinate for any $\alpha \in (0, \infty)$, as long as $\phi_\pi > \varphi(\Theta)$. In other words, for $\Theta \in (0, \Theta^\star]$, we have $\alpha^\star(\Theta) = \infty$. However, if $\Theta > \Theta^\star$, $\phi_\pi > \varphi(\Theta)$ is no longer sufficient for local determinacy. Local determinacy requires $\phi_\pi > \varphi(\Theta)$ alongside a small enough α: $\alpha < \alpha^\star(\Theta)$, i.e., a rule which is also backward-looking enough. □

D.2 Multiple steady states

Suppose that $\phi_\pi > \varphi(\Theta)$. Then, the stationary points of the economy are not affect by changing the policy rule from (6) to the inertial policy rule (26), which we study in Section 4.2. With the AIT policy rule, the steady state is represented by three nullclines, which can be written as:

$$\begin{aligned} 0 &= \phi(\pi^b - \pi^\star) - (\pi - \pi^\star) + \sigma(e^{-\gamma\Theta x} - 1) \\ 0 &= \rho(\pi - \pi^\star) - \kappa\left(e^x - 1\right) \\ 0 &= \alpha\left(\pi - \pi^b\right) \end{aligned}$$

Since the third nullcline implies that $\pi = \pi^b$, the first two nullcines are the same as in our baseline. Thus, the exact value of α does not affect the level of output and inflation in the untargeted steady state, but it does affect the stability properties. The steady state value of x in the untargeted steady state is still defined by the same equation as in the baseline model:

$$F(x) = \frac{\kappa\left(\phi_\pi - 1\right)}{\rho}(\mathrm{e}^x - 1) + \sigma\left(\mathrm{e}^{-\gamma\Theta x} - 1\right),$$

and the same argument as in Appendix B.3 establishes the existence of the untargeted steady state. □

D.3 Local stability of the untargeted steady state

Appendix B.3 showed that the untargeted steady state continues to exist even with the inertial rule, and that for a given ϕ_π, the output-gap $\underline{x}$ in the untargeted steady state is the same as in the baseline model, i.e., $\frac{dx}{d\alpha} = 0$. In what follows, we always impose $\phi_\pi > \varphi(\Theta)$ since it is a necessary condition for local determinacy of the targeted steady state under the inertial rule.

The Jacobian of the system (27)-(29) evaluated at the untargeted steady state can be written as:

$$A_{\underline{x}} = \begin{bmatrix} -\sigma\gamma\Theta e^{-\gamma\Theta\underline{x}} & -1 & \phi_\pi \\ -\kappa e^{\underline{x}} & \rho & 0 \\ 0 & \alpha & -\alpha \end{bmatrix}$$

Since we now have one predetermined-variable π^b and two jump-variables x, π^g, for the untargeted steady state to be locally determinate, we need one negative root, and two roots with positive real parts. This would then ensure that for a given π_0^b in the neighborhood of the untargeted steady state, there exists a unique (x_0, π_0^g) such that the trajectory $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ remains bounded. If, for a givne π_0^b, there exist multiple (x_0, π_0^g) for which the trajectory $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ remains bounded, then the untargeted steady state is locally indeterminate. Next, we show that for any given Θ, the untargeted steady state is locally indeterminate as $\phi_\pi > \varphi(\Theta)$, regardless of the magnitude of α.

The characteristic polynomial of the $A_{\underline{x}}$ is given by:

$$\mathcal{P}(z) \quad = \quad a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

where

$$\begin{aligned}
a_0 &= -1 \\
a_1 &= -\sigma\gamma\left(\Theta e^{-\gamma\Theta\underline{x}} - \Theta^\star\right) - \alpha \\
a_2 &= \kappa e^{\underline{x}}\left(1 + \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}\right) - \alpha\sigma\gamma\left(\Theta e^{-\gamma\Theta\underline{x}} - \Theta^\star\right) \\
a_3 &= -\alpha\kappa e^{\underline{x}}\left[\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}\right]
\end{aligned}$$

The stability of the system is governed by the pattern of sign changes in the sequence:

$$a_0, \qquad a_1, \qquad -\frac{a_0 a_3 - a_1 a_2}{a_1}, \qquad a_3$$

For the Jacobian to have one negative root and two roots with positive real parts, we need the sequence to have 2 sign changes. Recall from (b.3) in Appendix B.4, that if $\phi_\pi > \varphi(\Theta)$, then we have:

$$\phi_\pi - 1 - \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}} < 0$$

This implies that the fourth term on the sequence is $+$. Clearly, the sign of the first term in the sequence is $-$. If $\alpha > \sigma\gamma\left(\Theta^\star - \Theta e^{-\gamma\Theta\underline{x}}\right)$, then the sign of the second term in the sequence is $-$. So we have: $-,-,?,+$, and no matter what the sign of the third term is, we cannot have two sign changes. Thus, with large α, we have 2 negative and 1 positive root, which implies that for a given π^b, there are multiple bounded trajectories in the neighborhood of the untargeted steady state which converge to it.

Now consider the case in which $\alpha \leq \sigma\gamma\left(\Theta^\star - \Theta e^{-\gamma\Theta\underline{x}}\right)$. With small α, the sequence of signs is now $-,+,?,+$. So if the third term is negative, then a small α can ensure that the untargeted steady state is locally determinate. However, this is not the case, and the third term is positive:

$$\frac{\alpha\kappa e^{\underline{x}}\left(\phi^\diamond - \phi_\pi\right) + \sigma\gamma\left[\Theta^\star - \frac{\alpha}{\sigma\gamma} - \Theta e^{-\gamma\Theta\underline{x}}\right]\left\{\kappa e^{\underline{x}}\phi^\diamond + \alpha\sigma\gamma\left(\Theta^\star - \Theta e^{-\gamma\Theta\underline{x}}\right)\right\}}{\sigma\gamma\left(\Theta^\star - \frac{\alpha}{\sigma\gamma} - \Theta e^{-\gamma\Theta\underline{x}}\right)} > 0,$$

where $\phi^\diamond = 1 + \frac{\rho\sigma\gamma\Theta}{\kappa} e^{-(1+\gamma\Theta)\underline{x}}$, and we have used the fact that if $\phi_\pi > \varphi(\Theta)$, then $\phi_\pi < \phi^\diamond$. Furthermore, we know that $\Theta^\star \geq \Theta e^{-\gamma\Theta\underline{x}} + \frac{\alpha}{\sigma\gamma} > \Theta e^{-\gamma\Theta\underline{x}}$ in this case. So even with small α, the sequence is $-,+,+,+$, which only has one sign change. Thus, regardless of the magnitude of α, the untargeted steady state is locally indeterminate as long as $\Theta > 0$ and $\phi_\pi > \varphi(\Theta)$. □

D.4 Global indeterminacy

Appendix D.2 showed that as long as $\Theta > 0$, the untargeted steady state always exists. Furthermore, Appendix D.3 showed that this untargeted steady state is always locally indeterminate, as long as

$\phi_\pi > \varphi(\Theta)$, regardless of the magnitude of α. Consequently, there is always global indeterminacy. This is because local determinacy of the untargeted steady state implies that for a given π_0^b in the neighborhood of the untargeted steady state there exists multiple (x_0, π_0^g) such that there are at least two trajectories $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ which remain bounded forever. In fact, since the untargeted steady state has two negative and one positive eigenvalue, there exists a 2 dimensional stable manifold containing the untargeted steady state. Any trajectory which originates in this stable manifold remains bounded, in fact it converges to the untargeted steady state. Consequently, there is at least one π_0^b, for which there exists multiple (x_0, π_0) such that the trajectories $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ satisfy equilibrium and always remain bounded. There are even more bounded trajectories which start close to the targeted steady state, as Proposition 7 shows.

D.4.1 Proof of Proposition 7

For a given $\Theta > 0$, the Jacobian of the system (27), (28), (29), evaluated at the targeted steady state, can be written as:

$$A = \begin{bmatrix} -\sigma\gamma\Theta & -1 & \phi_\pi \\ -\kappa & \rho & 0 \\ 0 & \alpha & -\alpha \end{bmatrix}$$

The trace of A can be written as:

$$tr(A) = -\sigma\gamma\Theta + \rho - \alpha = -\sigma\gamma(\Theta - \Theta^\star) - \alpha,$$

where we have used the definition $\Theta^\star = \frac{\rho}{\sigma\gamma}$. Next, the determinant of A is given by:

$$det(A) = -\alpha\kappa\Big(\phi_\pi - \varphi(\Theta)\Big),$$

Since we maintain that $\phi_\pi > \varphi(\Theta)$, we know that $det(A) < 0$ for any $\alpha \in (0, \infty)$. We need to show that $\exists\alpha$ for which the two complex roots have zero real parts. To prove this, we use Orlando's formula, which can be written as:[44]

$$H = -det(A) + tr(A) \times G(A),$$

where $G(A)$ denotes the pairwise product of the eigenvalues of the matrix A and is given by:[45]

$$G(A) = \begin{vmatrix} -\sigma\gamma\Theta & -1 \\ -\kappa & \rho \end{vmatrix} + \begin{vmatrix} -\sigma\gamma\Theta & \phi_\pi \\ 0 & -\alpha \end{vmatrix} + \begin{vmatrix} \rho & 0 \\ \alpha & -\alpha \end{vmatrix} = \alpha\sigma\gamma\left(\Theta - \Theta^\star\right) - \kappa\varphi(\Theta)$$

Using these expressions, we can write H as:

$$H = \alpha\kappa\Big(\phi_\pi - \varphi(\Theta)\Big) + \left[\sigma\gamma\left(\Theta - \Theta^\star\right) + \alpha\right]\left[\kappa\varphi(\Theta) - \alpha\sigma\gamma\left(\Theta - \Theta^\star\right)\right],$$

[44] See pp. 196-198 in Chapter XV of Gantmacher (1960).

[45] Let z_1, z_2 and z_3 denote the three eigenvalues of the matrix A. Then $tr(A) = z_1 + z_2 + z_3$, $det(A) = z_1 z_2 z_3$ and $G(A) = z_1 z_2 + z_2 z_3 + z_3 z_1$.

which can be further simplified to

$$H = \kappa \times \alpha\Big(\phi_\pi - \psi(\alpha)\Big),$$

where $\psi(\alpha)$ is defined in (d.1). Notice that $H = 0$ describes values of α (if one exists) for which two of the eigenvalues cancel each other out. $H = 0$ requires that either (i) two of the eigenvalues are purely imaginary or (ii) two real eigenvalues have the same magnitude but opposite sign. Since Appendix D.1 showed that for $\Theta > \Theta^\star$, as long as $\alpha < \alpha^\star(\Theta)$, there are two complex roots with positive real parts and one real negative root. Thus, it follows that at $\alpha = \alpha^\star(\Theta)$ (which is defined in Appendix D.1), the two complex roots are purely imaginary and thus cancel each other out, leaving the trace to equal the remaining negative root. Thus, a Hopf bifurcation occurs at $\alpha = \alpha^\star(\Theta)$. While verifying that the first Lyapunov coefficient of this 3 dimensional system is possible, it is extremely cumbersome and we verify numerically that it is negative, implying that the Hopf bifurcation is supercritical. Consequently, the Hopf bifurcation theorem implies that for any $\Theta > 0$, $\exists \underline{\alpha}(\Theta) < \alpha^\star(\Theta)$, such that for $\alpha \in (\underline{\alpha}(\Theta), \alpha^\star(\Theta)]$, all trajectories (except those which begin on the one dimensional stable manifold around the targeted steady state), which originate near the targeted steady state converge to a stable cycle. In this region, since $\alpha < \alpha^\star(\Theta)$, Proposition 6 implies that the targeted steady state is locally determinate. However, since for a given π_0^b in the neighborhood of the targeted steady state, there exists multiple (x_0, π_0^g) such that the trajectories $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ remain bounded forever, there is global indeterminacy. The convergence to a stable cycle is depicted graphically in Figure 6a, where we have set $\Theta = 28.1$ and $\alpha = 9$. Since $\Theta = 28.1 < \Theta^\star = 31.1$ under our calibration, $\alpha^\star(28.1) = \infty$ and $\underline{\alpha}(28.1) \approx 1.03$ under our calibration. Thus, for any $\alpha > 1.03$, any trajectory originating near the targeted steady state converges to a stable cycle.

Next, Theorem 7.2 and Corollary 7.1 of Kopell and Howard (1975) (which are generalizations of Theorem 2 to $n > 2$ dimensions) imply that for ϕ_π close to $\varphi(\Theta)$, there exists a homoclinic orbit at the boundary of the stable cycles mentioned above. Thus, for $\alpha = \underline{\alpha}(\Theta)$, there exists a homoclinic orbit which cycles around the targeted steady state, while passing through the untargeted steady state. The homoclinic orbit is depicted graphically in Figure 6b. Even in this case, since $\underline{\alpha}(\Theta) < \alpha^\star(\Theta)$, Proposition 6 shows that the targeted equilibrium is locally determinate. However, there is still global indeterminacy, since for any π_0^b in the neighborhood of the targeted steady state, all combinations of (x_0, π_0) in the neighborhood of the targeted steady state are such that the trajectories $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ remain bounded forever. There is a unique one dimensional manifold around the targeted steady state along which the trajectories converge to the targeted steady state and remain bounded, while all trajectories originating off this stable manifold converge to the homoclinic orbit. At the bifurcation point, $\alpha = \alpha^\star(\Theta)$, for a given π_0^b in the neighborhood of the targeted steady state, while the first-order terms do not move the system towards or away from the targeted steady state, the higher-order terms ensure that all trajectories $\{x_t, \pi_t^g, \pi_t^b\}_{t=0}^\infty$ converge to the targeted steady state, and hence remain bounded, implying global indeterminacy in this case as well.

Finally, for $\alpha < \underline{\alpha}(\Theta)$, there are no stable cycles and Proposition 6 implies that the targeted steady state is locally determinate. However, Theorem 7.2 and Corollary 7.1 of Kopell and Howard (1975) also guarantee that for $\alpha \in (0, \alpha^\star(\Theta))$, there exists a saddle connection along which the economy can transition from the neighborhood of the targeted steady state to the untargeted steady state. Furthermore, any trajectory starting on this saddle connection constitutes a bounded equilibrium since it converges to the untargeted steady state and remains bounded forever. Thus, even in the range $\alpha \in [0, \underline{\alpha}(\Theta))$, the equilibrium is globally indeterminate. □

E Proof of Proposition 8

This section contains the proof of claims relating to the model described by (32a) and (32b). Policy rule (31) ensures the existence of a unique bounded equilibrium in which the economy remains that the targeted steady state $(x, \pi^g) = (0, 0)$ at all dates. Any trajectories which start at $(x_0, \pi_0^g) \neq (0, 0)$ diverge from the targeted steady state and become unbounded. To show this, we first prove that the targeted steady state $(0, 0)$ is the only steady state, i.e., the policy rule (31) eliminates the untargeted steady state. To see this, we can set $\dot{x} = \dot{\pi}_t^g = 0$ in (32a) and (32b), and can describe steady state output-gap as the solution to a one-dimensional equation:

$$F(x) = \frac{(\phi_\pi - 1)\kappa}{\rho}(e^x - 1) = 0$$

As always, $x = 0$ clearly satisfies this equation, implying that the targeted steady state $x = \pi^g = 0$ still exists. In addition, we can write the derivative of $F(x)$ as:

$$F'(x) = \frac{(\phi_\pi - 1)\kappa}{\rho}e^x,$$

which, as long $\phi_\pi > 1$, is positive for all $x \in (-\infty, \infty)$. Thus, $F(x)$ only intersects the x-axis once at $x = 0$. Next, we show that as long $\phi_\pi > 1$, the targeted equilibrium is locally determinate. Given the policy rule (31), the dynamics of the output-gap and inflation-gap close to the targeted steady state $(0, 0)$ is given by:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t^g \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}}_{A} \begin{bmatrix} x_t \\ \pi_t^g \end{bmatrix}$$

Local stability of the system depends on the eigenvalues of the matrix A. The trace of the matrix A is given by $\rho > 0$, which implies that the sum of both eigenvalues is positive. The determinant of A is given by $\kappa(\phi_\pi - 1)$, and is always positive as long as $\phi_\pi > 1$. This implies that the product of the two eigenvalues is also positive. Thus, for any $\Theta > 0$, as long as $\phi_\pi > 1$, both eigenvalues are positive and real, implying that $(0, 0)$ is unstable (locally determinate). Finally, global determinacy follows from the fact that under the policy rule (31), the dynamics of x, π^g are given by (32a)-(32b), which are the same as in the RANK benchmark $\sigma > 0$. Thus, the claim follows from the proof in Appendix B.1. □